TRIANGULAR FUZZY METRIC

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ملخص

يقدم هذا البحث طريقة جديدة لترتيب الأعداد المشوشة (الفزية) المثلثية وكذلك قياس جديد للمسافة بين النقاط في المجموعات المشوشة (الفزية).

ABSTRACT

This paper presents a new method to rank triangular fuzzy numbers as well as a new metric (triangular fuzzy metric) on the set of fuzzy points. This metric can be used in both studying fuzzy topological spaces and decision-making theory.

INTRODUCTION

There have been several definitions and studies of fuzzy metric spaces. Kramosil and Michalek[4] introduced the fuzzy metric spaces by generalizing the concept of probabilistic metric space. Puri and Ralescu[5], Heilpren[1] and Kaleva[2] used the concept of the Husdorff metric by defining the distance between two fuzzy sets as the supremum of Hausdorff distances of their α-level sets. Kaleva and Seikkala[3] defined the distance between two points to be a non-negative fuzzy number. In

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this paper we will introduce the distance between two fuzzy points using a triangular fuzzy number. In section 3, an ordering will be defined on the set of triangular fuzzy numbers to obtain an inequality which is analogous to the ordinary triangle inequality.

1. NOTATION AND BASIC DEFINITIONS

<u>DEFINITION</u> <u>2.1</u> : A fuzzy number A is a fuzzy subset of the real line R with :y

1. The α-level of A ; A^{α} , is a convex set for each α in the interval (0,1]

2. A is an upper semicontinuous function.

3. A is normal , i.e there exists an r in R such that A(r) = 1.

A fuzzy number A is finite if the support of A; the set $\{x \in R : A(x) > 0\}$ is a finite interval. A is nonnegative if A(x)=0 for every negative real number x. A fuzzy number A is unimodal if the set $\{x : A(x)=1\}$ is a single point. According to the above definition, a unimodal finite fuzzy number A can be represented by the triple (a,b,c) where A is nondecreasing on the interval (a.b), nonincreasing on the interval (b,c). takes the value 1 at x = b and zero for $x \le a$ and $x \ge b$. If A is linear on (a,b) and (b,c) then unimodal finite fuzzy number is called a triangular fuzzy number. let G be the set of all triangular fuzzy numbers. The fuzzy number A=(b,b,b) is called a crisp fuzzy number.

The - α level of the fuzzy set A in G is a closed interval $[a_1, a_2]$ for each α in (0,1].

DEFINITION 2.2 : if $A = (a_1,a_2,a_3)$ and $B = (b_1,b_2,b_3)$ are triangualr fuzzy numbers then A+B is the triangular fuzzy number $(a_1+b_1 , a_2+b_2 , a_3+b_3)$.

DEFINITION 2.3 : A fuzzy set A in X is called a fuzzy poin if A (y) = 0 for all y in X except at one element. say a. If A (a) = r where $0 < r \leq 1$ then the fuzzy point will be denoted by a_r.

2. A TOTAL ORDER ON G

DEFINITION 3.1 : If X and Y are the triangular fuzzy numbers

(x_1,x_2,x_3), (y_1,y_2,y_3), respectively, we define:

$$T(X:Y) = \sum_{i=1}^{3} (x_i - \min\{x_2,y_2\})$$

In the following , we consider $A = (a_1,a_2,a_3)$, $B = (b_1,b_2,b_3)$ and $C = (c_1,c_2,c_3)$ to be any triangluar fuzzy numbers.

LEMMA 3.1: If $T(A:B) \leq T(B:A)$ and $T(B:C) \leq T(C:B)$ where $a_2 \leq b_2 \leq c_2$ then $T(A:C) \leq T(C:A)$.

Proof: $T(A:C) = a_1 + a_2 + a_3 - 3a_2 = T(A:B) \le T(B:A)$
$= (b_1 + b_2 + b_3 - 3a_2) = T(B:C) + 3b_2 - 3a_2$
$\le T(C:B) + 3b_2 - 3a_2 = c_1 + c_2 + c_3 - 3b_2 + 3b_2 - 3a_2$
$= c_1 + c_2 + c_3 - 3a_2 = T(C:A)$.

<u>LEMMA</u> <u>3.2</u>: If $T(A:B) \le T(B:A)$ and $T(B:C) \le T(C:B)$ where
$\le c_2 \le b_2$ Then $T(A:C) \le T(C:A)$.

Proof : $T(A:C) = a_1 + a_2 + a_3 - 3a_2 = T(A:B)$
$\le T(B:A) = b_1 + b_2 + b_3 - 3a_2$
$= b_1 + b_2 + b_3 - 3c_2 + 3c_2 - 3a_2$
$= T(B:C) + 3c_2 - 3a_2 \le T(C:B) + 3c_2 - 3a_2$
$= c_1 + c_2 + c_3 - 3c_2 + 3c_2 - 3a_2$
$= c_1 + c_2 + c_3 - 3a_2 = T(C:A)$

Similarly , we can prove the following four lemmas,

<u>LEMMA</u> <u>3.3</u>: If $T(A:B) \le T(B:A)$ and $T(B:C) \le T(C:B)$ where $b_2 \le a_2 \le c_2$ Then $T(A:C) \le T(C:A)$

<u>LEMMA</u> <u>3.4</u>: If $T(A:B) \le T(B:A)$ and $T(B:C) \le T(C:B)$ where $b_2 \le c_2 \le a_2$ then $T(A:C) \le T(C:A)$

<u>LEMMA</u> <u>3.5</u> : If $(T(A:B) \le T(B:A)$ and $T(B:C) \le T(C:B)$ where $c_2 \le a_2 \le b_2$ Then $T(A:C) \le T(C:A)$

LEMMA 3.6 : If $T(A:B) \leq T(B:A)$ and $T(B:C) \leq T(C:B)$ where $c_2 \leq b_2 \leq a_2$ Then $T(A:C) \leq T(C:A)$

DEFINITION 3.2: For any triangular fuzzy numbers A and B, we say that $A \leq_T B$ iff $T(A:B) \leq T(B:A)$ and $A =_T B$ iff $T(A:B) = T(B:A)$.

Now, lemmas 3.1 - 3.6 proves the following theorem:

THEOREM 3.1: For any triangular fuzzy numbers A, B and C. If $A \leq_T B$ and $B \leq_T C$ then $A \leq_T C$

THEOREM 3.2: The relation $\leq_T$ defined on the set of triangluar fuzzy numbers G is a total order.

Proof : It is obvious that $\leq_T$ both reflexive and antisymmetric.
Also, $\leq_T$ is transtive (by theorem 3.1).
Therefore . $\leq_T$ is a total order on G.

THEOREM 3.3: If $B=(b_1,b_2,b_3)$ and $A=(a_1,a_2,a_3)$ are two triangular fuzzy numbers such that $b_i \leq a_i$ for all i, then $T(B:A) \leq T(A:B)$ and therefore , $B \leq_T A$.

Proof: $T(A:B)-T(B:A) = \Sigma a_i - 3b_2)-(\Sigma b_i-3b_2) = \Sigma(a_i - b_i) \geq 0$

Therefore, $T(B:A) \leq T(A:B)$ and $B \leq_T A$

3. FUZZY TRIANGULAR METRIC:

We will now define a fuzzy metric on the set of fuzzy points of a set X.

DEFINITION 4.1 : let x_s and y_r be twp fuzzy points. Let a = min {r,s} and b=max {r,s}.

For a fixed number K where $0 < K \leq 1/2$ we define the K-distance between the points x_s, y_r to be the triangular fuzzy number $d(x_s,y_r)$ where the α - level of $d(x_s,y_r)$ is equal to

$$\overset{\alpha}{d}(x_s,y_r) = \begin{cases} [L,R] \text{ if } x \neq y \\ [D,M] \text{ if } x = y \end{cases}$$

where $L = a + (1 - K)(b - a)\alpha$
$R = a + (1 - \alpha K)(b-a)$
$D = \alpha (1 - K) |s-r|$
$M = (1 - \alpha K) |s-r|$
$a = \min \{s,r\}$, $b = \max \{s,r\}$

Remark : It is easy to verify that the fuzzy number whose α-level defined in Definition 4.1 is the triangular fuzzy number $d(x_s, y_r)$ where $d(x_s, y_r) = (a, Ka + (1 - K)b , b)$ if $x \neq y$ and equal to $(0, (1 - K) | s - r | , | s - r |)$ if $x = y$

Now we will show that d is a metric.

In the following we consider arbitrary fuzy points P, Q and W where $P = x_s$, $Q = y_r$, $W = z_t$, s,r,t ε (0,1]. We write ≤ to mean $\leq_T$.

THEOREM 4.1 : If $P = x_s$, $Q = y_r$ and $W = z_t$ are any three fuzzy points where x,y and z are all distinct elements in a set X and s,r and t are elements in (0,1]. Then d(P,W) ≤ d(P,Q) + d(Q,W)

Proof : s,r and t are real numbers in the interval (0,1]. To prove this theorem we consider the following cases:

Case 1 : If s ≤ t ≤ r. Then d(P,W) = (s , Ks+Mt, t) and d(P,Q) + d(Q,W) = (s+t, K(s+t) + 2Mr, 2r). Since s ≤ s+t, Ks+Mt ≤ K(s+t) + 2Mr and t≤2r. Therefore, (by theorem 3.3) d(P,W) ≤ d(P,Q)+d(Q,W)

Using similar argument we can prove the following three cases:

Case 2 : If s ≤ r ≤ t

Case 3 : If t ≤ s ≤ r

Case 4 : If t ≤ r ≤ s

Case 5 : If $r \leq s \leq t$. Then $d(P,W) = (s, Ks+Mt, t)$ and $d(P,Q) + d(Q,W) = (2r , 2Kr+M(s+t), t+s)$. Consider the quantity $\{2Kr+ M(s+t)\} - \{Ks + Mt\} = 2Kr + Ms - Ks = 2Kr + s -2Ks \geq 2Kr\geq 0$ Since $s \geq 2Ks$. i.e $Ks+Mt \leq 2Kr + M(s+t)$. Therefore,
$T(d(P,Q) + d(Q,W) : d(P,W)) - T(d(P,W) : d(P,Q) + d(Q,W))$
$= 2r + 2Kr + M(s+t) +t +s -3(Ks + Mt) - (t+Ks+Mt + s) +3 (Ks + Mt)$
$= 2r + 2Kr + s - 2Ks \geq 2r + 2Kr \geq 0$ since $s \geq 2Ks$. Therefore,
$T(d(P,W) : d(P,Q)+d(Q,W)) < T(d(P,Q)+d(Q,W)):d(P,W)$
i.e (by Definition 3.2) $d(P.W) < d(P,Q) + d(Q,W)$

Case 6 : If $r \leq t \leq s$. Then $d(P,W) = (t, Kt+Ms, s)$ and $d(P,Q) + d(Q,W) = (2r , 2Kr+M(s+t) , s+t)$ using the same argument in case 5 , we have : $d(P,W) \leq d(P,Q) + d(Q,W)$. and the proof of this theorem is complete.

Now, In theorem 4.1 if the assumption that x,y and z are all distinct is replaced by x,y and z are all equal then we have the following theorem :

<u>THEOREM</u> <u>4.2</u> : If $P = x_s$, $Q = y_r$ and $W = z_t$ are any three fuzzy points where x,y, and z are all equal elements in a set X. and s,r and t are elements in (0.1]. Then , $d(P,W) \leq d(P,Q) + d(Q.W)$

Proof : Since s,r and t are real numbers in the interval (0,1],

we have to consider the following cases in order to prove the theorem.

Case 1 : If $s \le r \le t$. Then $d(P,W) = (0, M(t-s), t-s)$ and $d(P,Q)+d(Q,W)=(0, M(t-s),t-s)$ which means $d(P,W) = d(P,Q)+ d(Q,W)$ i.e $d(P,W) \le d(P,Q) + d(Q,W)$

Case 2 : If $s \le t \le r$. Then $d(P,W) = (0 , M(t-s), t-s)$ and $d(P,Q)+d(Q,W) = (0,M(2r-t-s), 2r-t-s)$ we have $M(t-s) \le M(2r-t-s)$ since $2r-t \ge t$, and $t-s \le 2r-t-s$ since $2r-t \ge t$. Therefore. (by theorem 3.3) $d(P,W) \le d(P,Q) + d(Q,W)$

The remaining cases can be proved in a similar manner.

Case 3 : If $r \le s \le t$

Case 4 : If $r \le t \le s$

Case 5 : If $t \le s \le r$

Case 6 : If $t \le r \le s$

It remains to consider theorems 4.1 and 4.2 under the assumption that not all x,y and z are distinct.

<u>LEMMA</u> <u>4.1</u> : If $P=x_s$, $Q=y_r$ and $W=z_t$ are any three fuzzy points where $x = y \ne z$ and s,r and t are elements in (0,1]. Then , $d(P,W) \le d(P,Q) + d(Q,W)$

Proof : Since s,r and t are in (0,1), we have to consider the following possibilities:

(1) $r \le s \le t$: $d(P,W) = (s, Ks+Mt, t)$ and $d(P,Q) + d(Q,W) =$ $(r, Kr+Mt+M(s-r), t+s-r)$. We have : $(Kr + Mt + M(s-r))-(Ks+Mt)$ $= 2Kr - 2Ks + s - r = (s-r)(1-2K) \ge 0$ since $r \le s$ and $2K \le 1$.
Therefore, $Kr + Mt + M(s-r) \ge Ks + Mt$ and
$T (d(P,Q) + d(Q,W): d(P,W)) - T (d(P,W) : d(P,Q) + d(Q,W))$
$= \{r + Kr + Mt + M(s-r) + t + s - r - 3(Ks + Mt)) - (s + t - 2Ks - 2Mt)$
$= (s-r)(1-2K) \ge 0$ since $r \le s$, $2K \le 1$. Therefore,
$T (d(P,W) : d(P,Q) + d(P,W)) \le T (d(P,Q) + d(Q,W): d(P,W))$
i.e (by definition 3.2) $d(P.W) \le d(P,Q) + d(Q.W)$

(2) $r \le t \le s$: $d(P,Q)+d(Q,W) = (r, Kr+Mt+M(s-r), s+t-r)$ and $d(P,W) = (t, Kt+Ms, s)$. We have $Kr + Mt + M(s-r) - Kt - Ms =$
$= 2kr-2kt+t-r = (t-r)(1-2K) \ge 0$ since $r \le t$, $2K \le 1$.Therefore.
$T (d(P,Q) + d(Q,W): d(P,W) - T (d(P,W): d(P,Q) + d(Q,W))=$
$= \{r+Kr+Mt+M(s-r)+s+t-r-3(Kt+Ms)\}-(t+s-2(Kt+Ms\} = (t-r)(1-2K) \ge 0$ since $r \le t$, $2K \le 1$. Therefore, $T(d(P,W):d(P,Q)+d(Q,W)) \le$ $T (d (P,Q) + d(Q,W) : d(P,W))$ and $d(P,W) \le d(P,Q) + d(Q,W)$

In a similar mannar we can prove the remaining cases.

(3) $t \le s \le r$

(4) $t \le r \le s$

(5) $s \leq t \leq t$

(6) $s \leq t \leq r$

This completes the proof of the Lemma.

<u>LEMMA</u> <u>4.2</u> : If $P = x_s$, $Q = y_r$ and $W = z_t$ are any three fuzzy points where $x = z * y$ and s, r and t are elements in (0,1).
Then $d(P,W) \leq d(P,Q) + d(Q,W)$

Proof : Similar to our proof of lemma 4.1, we have to consider six cases:

(1) $s \leq r \leq t$:

Comparing the triangualr fuzzy number $d(P,W) = (0, M(t-s), t-s)$ with $d(P,Q) + d(Q,W) = (s+r, M(t+r) + K(r+s), t+r)$, Since $0 \leq s+r$, $M(t-s) \leq M(t+r) + K(r+s)$ and $t-s \leq t + r$. Therefore, $d(P,W) \leq d(P,Q) + d(Q,W)$.

The remaining cases can be proved in a similar manner.

(2) $s \leq t \leq r$

(3) $r \leq s \leq t$

(4) $r \leq t \leq s$

(5) $t \leq s \leq r$

(6) $t \leq r \leq s$

<u>LEMMA</u> <u>4.3</u> : If $P = x_s$, $Q = y_r$ and $W = z_t$ are any three fuzzy points where $y = z * x$ and s,r and t are elements in (0,1], then
$d(P,W) < d(P,Q) + d(Q.W)$

Proof : Similar to our proof of Lemma 4.1 and 4.2 , we have to consider six cases :

(1) $r \leq s \leq t$: $d(P,Q) + d(Q,W) = (r, Kr+Ms+M(t-r), s+t-r)$ and $d(P,W) = (s, Ks + Mt ,t)$. Since $(Kr +Ms +m(t-r) - (Ks+Mt) =(s-r)(1-2K) \geq 0$. Therefore, $Ks + Mt \leq Kr +Ms +M(t-r)$ and $T (d(P,Q) + d(Q,W)) : T (d(P,W) - T(d(P,W) :d(P,Q) +d(Q,W) = r+Kr+Ms+M(t-r)+s+t-r-3(Ks+Mt) - (s+t -2Ks - 2Mt) = (s-r)(1-2K) \geq 0$ since $r \leq s$, $2K \leq 1$ which implies that $d(P,W) \leq d(P,Q) + d(Q,W)$.

(2) $r \leq t \leq s$: $d(P,W) = (t, Kt+Ms , s)$ and $d(P,Q) + d(Q,W) = (r, Kr +Ms +M(t-r) , s+t-r)$. Since $(Kr +Ms +M(t-r) - (Kt+Ms) = (t-r)(1-2k) \geq 0$. Therefore, $Kt + Ms \leq Kr +Ms +M(t-r)$ and $T (d(P,Q) + d(Q,W) : d(P,W)) - T (d(P,W): d(P,Q) + d(Q,W)) =$
$= r+Kr+Ms+M(t-r)+s+t-r-3(Kt+Ms)-(t+s-2(Kt+Ms))= (t-r)(1-2K) \geq 0$ since $r \leq t$, $2K \leq 1$ which implies that $d(P,W) \leq d(P,Q) + d(Q,W)$

Similarly we can prove the remaining four cases.

(3) $s \leq r \leq t$

(4) $s \leq t \leq r$

(5) $r \leq s \leq t$

(6) $r \leq t \leq s$

This completes the proof of this Lemma.

Now we are ready to prove the following theorem.

<u>THEOREM</u> <u>4.3</u> : For a fixed number K in (0,1/2], the K-distance defined on the set of fuzzy points of X (Definition 4.1) forms a metric.

Proof :

1. $d(x_s, y_r) \geq 0=(0,0,0)$ (The Zero fuzzy number). This follows directly from the definition.

2. If $x_s = y_r$ then $x = y$, $s = r$ and $d(x_s, y_r) = \bar{0}$
If $d(x_s, y_r) = 0$ then $x = y$ (Otherwise if $x \neq y$ then, since $\max\{s,r\} > 0$, $d(x_s, y_r) \neq 0$). Also, $|s-r| = 0$ i.e $s = r$. Therefore $X_s = Y_r$.

3. $d(x_s,y_r) = d(y_r,x_s)$. This is abvious from the definition

4. For any fuzzy points P,Q and W where $P = x_s$, $Q = y_r$ and $W=z_t$. We have $d(P,W) \leq d(P,Q) + d(Q,W)$. This follows from Lemma 4.1, 4.2, 4.3 and theorems 4.1, 4.2 when we considered all possibilities of equality on the elements x,y and z and the order of s,r and t on the interval (0,1]. Therefore. d is a metric. it is called the triangular fuzzy metric.

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