Fourier Transforms and Sampling

1 Definitions and Basic Ideas

- Given a continuous-time signal $x(t)$, its Fourier transform is defined as

$$X^f(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt. \tag{1}$$

- If the system with impulse response $x(t)$ is BIBO stable, i.e., if the poles of $X(s)$ are in the left half plane, then $X^f(\omega)$ is simply $X(s)$ evaluated at $s = j\omega$, i.e., $X^f(\omega) = X(j\omega)$. This is not true if $X(s)$ has poles on the imaginary axis as the examples below will show.

- Given $X^f(\omega)$, $x(t)$ can be obtained using the inverse Fourier transform which is defined as

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^f(\omega)e^{j\omega t}d\omega. \tag{2}$$

- The Fourier transform $X^f(\omega)$ describes the frequency content of the signal $x(t)$ and provides a representation of the signal as a combination of complex sinusoids (this representation is the inverse Fourier transform).

- Consider the signal $x(t) = e^{j\omega_0 t}$. This signal has only one frequency component (at frequency ω_0). Hence, we expect that the Fourier transform should be nonzero only for frequency ω_0. In fact, it can be shown that $X^f(\omega) = 2\pi\delta(\omega - \omega_0)$. This can be verified by computing the inverse Fourier transform of $2\pi\delta(\omega - \omega_0)$. Using the definition of inverse Fourier transform, the inverse Fourier transform of $2\pi\delta(\omega - \omega_0)$ is $\frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}d\omega = e^{j\omega_0 t}$ which implies that the Fourier transform of the signal $e^{j\omega_0 t}$ is $2\pi\delta(\omega - \omega_0)$.

- If we have a system with impulse response $h(t)$, then $H^f(\omega)$ describes the frequency content of the system in the sense that it describes what the system does (in terms of scaling and phase shift) to a sinusoidal input signal at frequency ω.

Example: The Fourier transform of $\delta(t)$ is 1.

Example: Consider $x(t) = e^{-at}u(t)$ with $a > 0$ being a constant. Then, the pole of $X(s) = \frac{1}{s+a}$ is in the left half plane. Hence, $X^f(\omega) = \frac{1}{j\omega+a}$.

Example: Consider $x(t) = \sin(\omega_0 t)$. Since the corresponding $X(s)$ has poles on the imaginary axis, we cannot find the Fourier transform of $x(t)$ by simply substituting $s = j\omega$ in $X(s)$. Instead, since $\sin(t) = \frac{1}{2j}(e^{j\omega_0 t} - e^{-j\omega_0 t})$, we find using the Fourier transform of $e^{j\omega_0 t}$ which was computed above that the Fourier transform of $\sin(t)$ is $\frac{1}{2j}[2\pi\delta(\omega - \omega_0) - 2\pi\delta(\omega + \omega_0)] = \frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$. Similarly, the Fourier transform of $\cos(\omega_0 t)$ can be shown to be $\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$.

2 Properties of the Fourier Transform

- Using the properties of the Laplace transform that were derived in previous lecture notes, we can infer corresponding properties of the Fourier transform. For example, if the Fourier transform of $x(t)$ is $X^f(\omega)$, then the Fourier transform of the shifted signal $x(t - t_0)$ is $e^{-j\omega t_0}X^f(\omega)$, i.e., time shift corresponds to modulation in frequency domain.

- If the Fourier transform of $x(t)$ is $X^f(\omega)$, then the Fourier transform of the derivative of $x(t)$, i.e., of $\frac{dx(t)}{dt}$ is $j\omega X^f(\omega)$.

- If the Fourier transform of $x(t)$ is $X^f(\omega)$, then the Fourier transform of the integral of $x(t)$, i.e., of $\int_{-\infty}^{t} x(\tau)d\tau$ is $\frac{1}{j\omega}X^f(\omega) + \pi X^f(0)\delta(\omega)$. Note that this property is slightly different from the corresponding property for the Laplace transform (recall that the Laplace transform of $\int_{-\infty}^{t} x(\tau)d\tau$ was shown to be $\frac{1}{s}X(s)$). Here, in the case of the Fourier transform, we have the additional term $\pi X^f(0)\delta(\omega)$. Intuitively, this term can be thought of as resulting from the DC (i.e., zero frequency or average) component of $x(t)$.

  Example: To find the Fourier transform of the signal $u(t)$, we write $u(t) = \frac{1}{2} + [u(t) - \frac{1}{2}]$. Noting that $\frac{1}{2} = \frac{1}{2}e^{j0t}$, the Fourier transform of $\frac{1}{2}$ is seen to be $\frac{1}{2}2\pi\delta(\omega) = \pi\delta(\omega)$. The signal $x_2(t) = [u(t) - \frac{1}{2}]$ has zero average value (since it is $+\frac{1}{2}$ for all positive t and $-\frac{1}{2}$ for all negative t). Since the derivative of $x_2(t)$ is equal to $\delta(t)$, we infer that $j\omega X_2^f(\omega) = 1$.

Since the average value of $x_2(t)$ is zero, this implies that $X_2^f(\omega) = \frac{1}{j\omega}$. Therefore, the Fourier transform of $u(t)$ is $\pi\delta(\omega) + \frac{1}{j\omega}$.

Example: Figure 1 shows a rectangular *pulse* signal $x(t)$ which is 1 for t between $-T$ and T (where T is a positive constant) and zero for $|t| > T$. The Fourier transform of $x(t)$ is given by $X^f(\omega) = \int_{-T}^{T} e^{-j\omega t} dt = 2\frac{\sin(\omega T)}{\omega} = 2T\text{sinc}(\frac{\omega T}{\pi})$. Recall that, as shown in Figure 2, sinc is defined as $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$. $X^f(\omega)$ is shown in Figure 3. Note that as T becomes smaller, the central lobe of $X^f(\omega)$ becomes wider, i.e., as the signal becomes narrower in time, its Fourier transform becomes broader in frequency. This is a standard observation which corresponds to the physical limitation that it is not possible to simultaneously *localize* in both time and frequency.

Example: Figure 4 shows the signal $x(t) = \frac{\sin(Wt)}{\pi t} = \frac{W}{\pi}\text{sinc}(\frac{Wt}{\pi})$ where W is a positive constant. It can be derived that its Fourier transform (shown in Figure 5) is a rectangular pulse which is 1 between $-W$ and W. Again, it can be seen that if the signal is made narrower in time by increasing W, then its Fourier transform becomes broader in frequency.

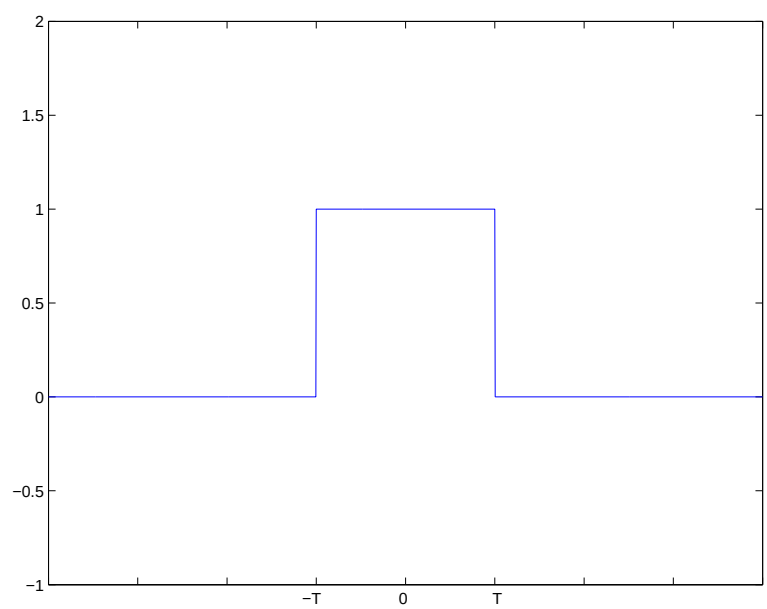

Figure 1: A rectangular pulse signal.

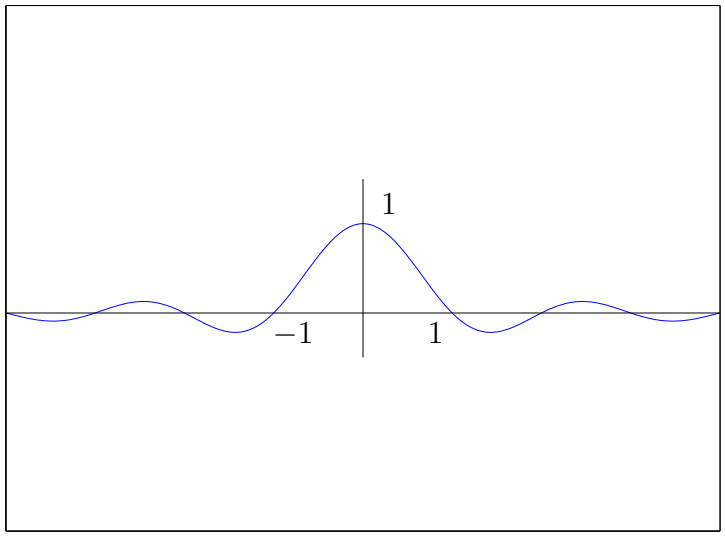

Figure 2: The sinc function.

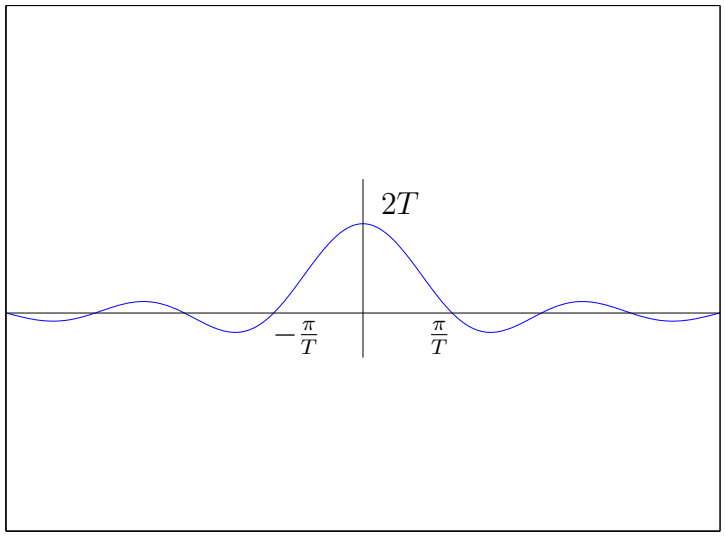

Figure 3: Fourier transform of a rectangular pulse signal.

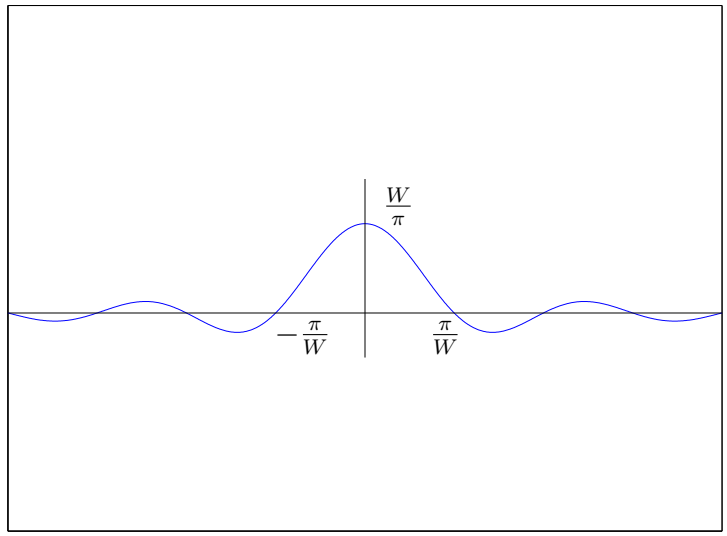

Figure 4: A sinc signal.

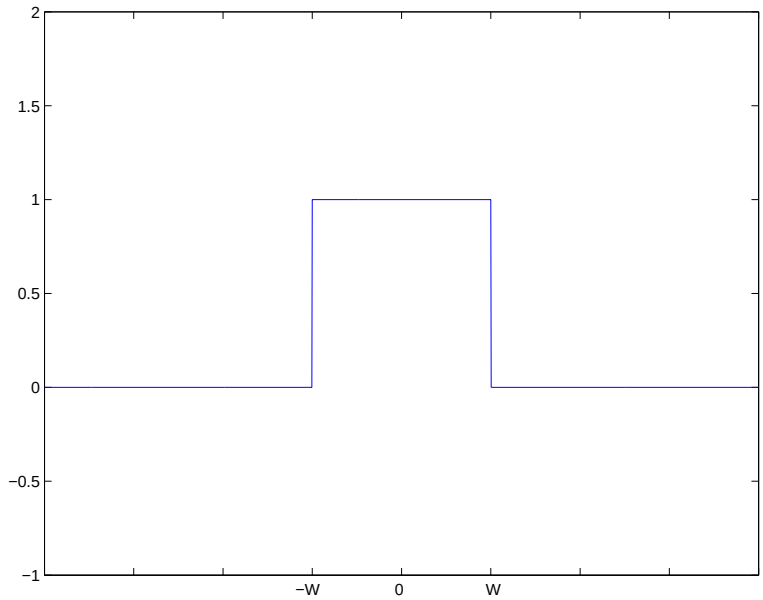

Figure 5: Fourier transform of a sinc signal.

- In the above examples, we saw that the Fourier transform of a rectangular pulse is a sinc function and the Fourier transform of a sinc function is a rectangular pulse. This is an example of the general *duality* theorem which can be stated as follows: If the Fourier transform of a signal $g(t)$ is $f(\omega)$, then the Fourier transform of $f(t)$ is $2\pi g(-\omega)$.

  Proof: If the Fourier transform of $g(t)$ is $f(\omega)$, then we have the equation $f(\omega) = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}dt$. Changing the dummy variables, we can equivalently write

$$f(t_1) = \int_{-\infty}^{\infty} g(\omega_1)e^{-jt_1\omega_1}d\omega_1 = \int_{-\infty}^{\infty} g(-\omega_1)e^{jt_1\omega_1}d\omega_1. \tag{3}$$

  Hence, $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[2\pi g(-\omega)]e^{jt\omega}d\omega$. Comparing with the definition of the inverse Fourier transform, this means that the inverse Fourier transform of $2\pi g(-\omega)$ is $f(t)$, i.e., the Fourier transform of $f(t)$ is $2\pi g(-\omega)$.

- **Parseval's Theorem:** As in the discrete-time case, it can be shown that the energy in the time domain is equal to the energy in the frequency domain, i.e., $\int_{-\infty}^{\infty}|x(t)|^2dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X^f(\omega)|^2d\omega$.

  Proof: The Parseval's theorem can be proved easily by noting the equations $|x(t)|^2 = x(t)x^*(t)$ and $x^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(X^f(\omega))^*e^{-j\omega t}d\omega$. Hence,

$$\begin{aligned}
\int_{-\infty}^{\infty}|x(t)|^2dt &= \int_{-\infty}^{\infty}x(t)\frac{1}{2\pi}\int_{-\infty}^{\infty}(X^f(\omega))^*e^{-j\omega t}d\omega dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}(X^f(\omega))^*\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dtd\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}(X^f(\omega))^*X^f(\omega)d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}|X^f(\omega)|^2d\omega. \qquad (4)
\end{aligned}$$

- If the signal $x(t)$ is periodic, then it is evident that it can have a frequency content at only some discrete frequencies. Specifically, if the period of a signal $x(t)$ is T, then it can contain components at frequencies $\frac{2\pi}{T}$, $\frac{4\pi}{T}$, and so on. In general, it can contain components at frequencies $\frac{2\pi k}{T}$ with $k = \ldots, -2, -1, 0, 1, 2, \ldots$. Hence, $x(t)$ can be represented in the form $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ where $\omega_0 = \frac{2\pi}{T}$ is the *fundamental*

frequency. The frequencies $k\omega_0$ are called *harmonics*. From this representation of $x(t)$, it is seen that $X^f(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$, i.e., $X^f(\omega)$ is comprised of a δ train with adjacent δ's separated by ω_0.

3 Sampling

Sampling is the process of extracting the values of a continuous-time signal at some particular discrete time instants generally given by nT_s with T_s being the *sampling period*. If we sample a continuous-time signal $x(t)$ with sampling period T_s, we get the discrete-time signal $y[n] = x(nT_s)$ as shown in Figure 6. The process of sampling is linear but time varying.

Figure 6: Sampling a continuous-time signal $x(t)$ to obtain a discrete-time signal $y[n]$.

Sampling can also be thought of as multiplication by a δ train. Recall the sampling property of a δ function which says that $x(t)\delta(t) = x(0)\delta(t)$, i.e., multiplication by $\delta(t)$ is equivalent to sampling the signal $x(t)$ at time 0. Hence, multiplication by a δ train is equivalent to sampling the signal at a sequence of evenly spaced time instants. This process is illustrated in Figure 7 and can be summed up as follows:

$$y(t) \;=\; x(t) \sum_{r=-\infty}^{\infty} \delta(t - rT_s) \;=\; \sum_{r=-\infty}^{\infty} x(rT_s)\delta(t - rT_s). \tag{5}$$

The frequency content (i.e., the Fourier transform) of the sampled signal $y(t)$ can be found using the Fourier transforms of $x(t)$ and $\sum_{r=-\infty}^{\infty} \delta(t - rT_s)$. To do this, the following two theorems are needed.

- If the Fourier transforms of any signals $x_1(t)$ and $x_2(t)$ are $X_1^f(\omega)$ and $X_2^f(\omega)$, respectively, then the Fourier transform of $x_1(t)x_2(t)$ is $\frac{1}{2\pi} X_1^f(\omega) * X_2^f(\omega)$.

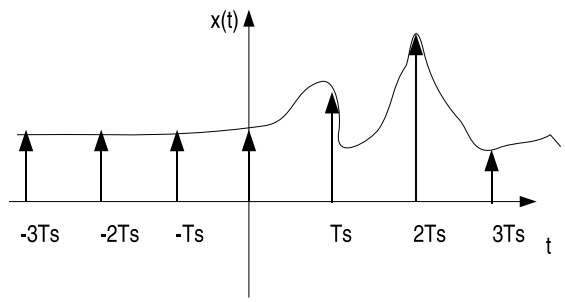

Figure 7: The sampling process.

Proof: This property of Fourier transforms can be proved analogously to the corresponding property for Laplace transforms.

- The Fourier transform of a δ train is a δ train, i.e., the Fourier transform of $\sum_{r=-\infty}^{\infty} \delta(t - rT_s)$ is $\frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi}{T_s}k)$

  Proof: The δ train $\sum_{r=-\infty}^{\infty} \delta(t - rT_s)$ is a periodic signal with period T_s. Hence, as shown earlier in this lecture notes, its Fourier transform must be of the form $\sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$ where $\omega_0 = \frac{2\pi}{T_s}$ and equivalently, we must have $\sum_{r=-\infty}^{\infty} \delta(t - rT_s) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$. Multiplying both sides of this equation by $e^{-jk_2\omega_0 t}$ with k_2 being any integer and integrating from $\frac{-T_s}{2}$ to $\frac{T_s}{2}$, we get

$$\int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} \sum_{r=-\infty}^{\infty} \delta(t - rT_s) e^{-jk_2\omega_0 t} dt \quad = \quad \int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jk_2\omega_0 t} dt. \quad (6)$$

  The left hand side of (6) is equal to $\sum_{r=-\infty}^{\infty} e^{-jk_2\omega_0 rT_s} \int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} \delta(t - rT_s) dt$. In the δ train of $\delta(t - rT_s)$, note that $\delta(t)$ is the only one in the interval $[-\frac{T_s}{2}, \frac{T_s}{2}]$. Hence, $\int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} \delta(t - rT_s) dt$ is 1 for $r = 0$ and 0 for any $r \neq 0$. Therefore, the left hand side of (6) is equal to $e^{-jk_2\omega_0 0T_s} = 1$. The right hand side of (6) is equal to $\sum_{k=-\infty}^{\infty} a_k \int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} e^{j(k-k_2)\omega_0 t} dt$. It can be verified that $\int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} e^{j(k-k_2)\omega_0 t} dt = 0$ if $(k - k_2) \neq 0$ and $\int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} e^{j(k-k_2)\omega_0 t} dt = T_s$ if $(k - k_2) = 0$. Therefore, the right hand side of (6) is equal to $a_{k_2} T_s$. Hence, $a_{k_2} T_s = 1$ which implies that $a_{k_2} = \frac{1}{T_s}$ for all k_2. Therefore, the Fourier transform of the δ train $\sum_{r=-\infty}^{\infty} \delta(t - rT_s)$ is $\sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T_s} \delta(\omega - k\omega_0)$

Using the two theorems above, the Fourier transform of $y(t)$ is

$$\begin{aligned} Y^f(\omega) &= \frac{1}{2\pi} X^f(\omega) * \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi}{T_s} k) \\ &= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X^f(\omega - \frac{2\pi}{T_s} k). \end{aligned} \tag{7}$$

Hence, sampling causes copies of the frequency content $X^f(\omega)$ shifted by $\omega_0 = \frac{2\pi}{T}$. Each copy is scaled by $\frac{1}{T_s}$ relative to the original frequency content. This is illustrated in Figures 8 and 9.

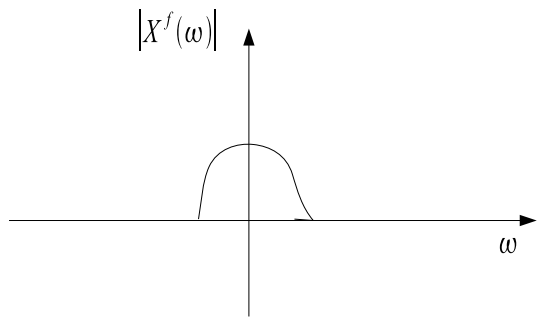

Figure 8: Frequency content of original signal.

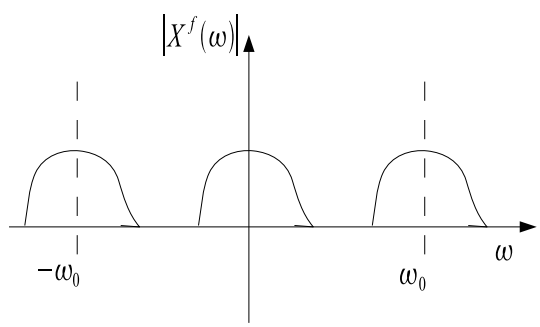

Figure 9: Frequency content of sampled signal.

If ω_0 is bigger than *twice the maximum frequency* (the *Nyquist rate*) in $x(t)$, then the adjacent copies of the frequency content $X^f(\omega)$ in $Y^f(\omega)$ are disjoint, i.e., do not overlap. In such a case, we can extract the frequency content of the original signal $x(t)$ using a low pass filter as illustrated in Figure 10. This means that if the sampling frequency is more than twice the maximum frequency in $x(t)$, then we can exactly reconstruct $x(t)$ if we are given the sampled data $y(t)$. As an example, if we had to transmit the signal

$x(t)$ from one place to another, we do not need to transmit the values of $x(t)$ at all times. Instead, it is sufficient to just send samples taken at discrete instants of time as long as our sampling frequency is large enough (i.e., bigger than twice the maximum frequency in $x(t)$). This fact is crucial in several applications of digital signal processing including digital communications.

Note: If we do not sample fast enough, then the adjacent copies of frequency content in $Y^f(\omega)$ overlap. In the event of such an overlap, it is not possible to reconstruct the original signal $x(t)$ from the sampled data. Also, the overlap causes the introduction of apparent lower frequencies, a phenomenon called *aliasing*.

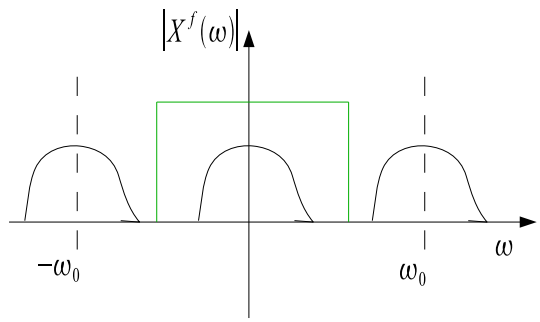

Figure 10: Reconstructing $x(t)$ using a low pass filter.

4 First and Second Order Low Pass Filters

The low pass filter depicted (in green) in Figure 10 is an ideal low pass filter. It has magnitude response exactly equal to 1 in the passband and exactly equal to 0 in the stopband. Furthermore, the transition between the passband and stopband is perfectly sharp. These ideal characteristics cannot be attained in practice. Any real-world design of a low pass filter would involve a tradeoff among these ideal characteristics. We can design the filter to have an almost flat magnitude response in the passband (a Butterworth filter) or we can design the filter to have a very sharp transition between passband and stopband (an Elliptic filter); but, both objectives cannot be attained simultaneously. As can be expected, if the order of the filters is allowed to be large, then these objectives can be attained more accurately. However, increasing the order of the filter implies an increased complexity in design and implementation and higher run-time computational and memory requirements — engineering is full of tradeoffs.

The two simplest low pass filters are the first and second order low pass filters:

- First order filter: $H(s) = \frac{1}{\tau s+1}$
- Second order filter: $H(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$

Exercise: Use the "bode" command in Matlab to plot the magnitude response and phase response of these filters for different values of the parameters (τ in the case of the first order filter; ζ and ω_n in the case of the second order filter) and get a feel for how the frequency response of the filters can be designed by varying the parameters. Observe also how reducing ζ in the second order filter makes the *resonance* peak in the magnitude response higher.