One-Sided Laplace Transform and Differential Equations

As in the discrete-time case, the one-sided transform allows us to take initial conditions into account.

1 Preliminaries

- The one-sided Laplace transform of a continuous-time signal $x(t)$ is defined as $X(s) = \int_0^\infty x(t)e^{-st}dt$.

- If the one-sided Laplace transform of $x(t)$ is known to be $X(s)$, then the one-sided Laplace transform of $\frac{dx(t)}{dt}$ is $sX(s) - x(0)$.

  Proof: By definition, the one-sided Laplace transform of $\frac{dx}{dt}$ is

$$\begin{aligned} \int_0^\infty \frac{dx(t)}{dt} e^{-st} dt &= x(t)e^{-st}|_0^\infty - \int_0^\infty x(t)(-se^{-st})dt \\ &= -x(0) + sX(s) \;=\; sX(s) - x(0) \end{aligned} \tag{1}$$

  by using integration by parts.

- Generalizing, the one-sided Laplace transform of $\frac{d^2x(t)}{dt^2}$ is

$$s[sX(s) - x(0)] - \dot{x}(0) \;=\; s^2X(s) - sx(0) - \dot{x}(0). \tag{2}$$

  This is obtained from the property above by recognizing that $\frac{d^2x(t)}{dt^2} = \frac{d}{dt}\left(\frac{dx(t)}{dt}\right)$. Similarly, the one-sided Laplace transforms of higher order derivatives can be written out in terms of $X(s)$. For instance, the one-sided Laplace transform of $\frac{d^3x(t)}{dt^3}$ is

$$s\Big[s[sX(s) - x(0)] - \dot{x}(0)\Big] - \ddot{x}(0) = s^3X(s) - s^2x(0) - s\dot{x}(0) - \ddot{x}(0). \tag{3}$$

- In the case of the two-sided Laplace transform which was considered in the previous lecture notes, we saw that if the two-sided Laplace transform of $x(t)$ is known to be $X(s)$, then the two-sided Laplace transform of $\frac{dx(t)}{dt}$ is $sX(s)$. In the case of the one-sided Laplace transform, we get an additional term depending on the initial condition as seen above.

- By default, if we mention "Laplace transform" without explicitly specifying if it is one-sided or two-sided, then we implicitly mean a two-sided Laplace transform.

2 Solving Differential Equations

Given a differential equation, an input signal, and initial conditions, we have two methods to solve the differential equation:

1. Using the one-sided Laplace transform
2. Using the method of guessing exponentials.

These two methods are explained below with examples.

2.1 Solving Differential Equations Using The One-Sided Laplace Transform

In this method, we take the one-sided Laplace transform of both sides of the given differential equation. This procedure will result in an equation involving $Y(s)$, $X(s)$, and the given initial conditions. We then solve for $Y(s)$ in terms of $X(s)$ and the initial conditions. Then, taking the inverse Laplace transform of $Y(s)$ gives us $y(t)$. We will only consider differential equations wherein initial conditions at $t = 0$ are given and we are required to find $y(t)$ for $t \geq 0$, i.e., we interpret the differential equation as a causal system. This means that we must take the causal inverse Laplace transform of $Y(s)$ to find $y(t)$.

Example: Solve the differential equation $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = x(t)$ given the input signal $x(t) = e^t u(t)$ and the initial conditions $y(0) = 1$, $\dot{y}(0) = 1$:

Taking one-sided Laplace transforms of both sides, we have

$$s^2Y(s) - sy(0) - \dot{y}(0) + 3sY(s) - 3y(0) + 2Y(s) \;=\; X(s). \tag{4}$$

Therefore,

$$Y(s) \;=\; \frac{X(s) + sy(0) + \dot{y}(0) + 3y(0)}{s^2 + 3s + 2} \;=\; \frac{\frac{1}{s-1} + s + 4}{s^2 + 3s + 2}$$

$$= \frac{1+(s-1)(s+4)}{(s^2+3s+2)(s-1)} = \frac{s^2+3s-3}{(s^2+3s+2)(s-1)} \tag{5}$$

where we have used the fact that the one-sided Laplace transform of $e^t u(t)$ is $\frac{1}{s-1}$. Decomposing into partial fractions, we get

$$Y(s) = \frac{0.1667}{s-1} + \frac{2.5}{s+1} + \frac{-1.6667}{s+2}. \tag{6}$$

Hence, for all $t \geq 0$,

$$y(t) = 0.1667e^t + 2.5e^{-t} - 1.6667e^{-2t}. \tag{7}$$

Example: Solve the differential equation $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = x(t)$ given the input signal $x(t) = e^{-t}u(t)$ and the initial conditions $y(0) = 1$, $\dot{y}(0) = 1$:

As in the previous example, taking one-sided Laplace transforms of both sides, we have

$$s^2Y(s) - sy(0) - \dot{y}(0) + 3sY(s) - 3y(0) + 2Y(s) = X(s). \tag{8}$$

Therefore,

$$\begin{aligned} Y(s) &= \frac{X(s) + sy(0) + \dot{y}(0) + 3y(0)}{s^2+3s+2} = \frac{\frac{1}{s+1} + s + 4}{s^2+3s+2} \\ &= \frac{1+(s+1)(s+4)}{(s^2+3s+2)(s+1)} = \frac{s^2+5s+5}{(s^2+3s+2)(s+1)} \end{aligned} \tag{9}$$

where we have used the fact that the one-sided Laplace transform of $e^{-t}u(t)$ is $\frac{1}{s+1}$. Decomposing into partial fractions, we get

$$Y(s) = \frac{2}{s+1} + \frac{1}{(s+1)^2} + \frac{-1}{s+2}. \tag{10}$$

Hence, for all $t \geq 0$,

$$y(t) = 2e^{-t} + te^{-t} - e^{-2t}. \tag{11}$$

2.2 Solving Differential Equations Using The Method of Guessing Exponentials

As discussed in the lecture notes on inverse Laplace transforms, we expect that $y(t)$ will include terms of the form $e^{p_i t}$ where $p_1, \ldots, p_n$ are the poles of the system (i.e., the poles of $H(s)$) and the poles of the input signal (i.e., the poles of $X(s)$). If any of the poles is repeated (i.e., has multiplicity greater than one), then we get additional terms. In general, if a pole p_i is repeated k times, then $y(t)$ includes the terms $e^{p_i t}u(t), \ldots, t^{k-1}e^{p_i t}u(t)$. Hence, by finding the poles of $H(s)$ and the poles of $X(s)$, we can guess the form of $y(t)$. To find the coefficients of the terms in the guessed form of $y(t)$, we use the initial conditions and the given differential equation. The procedure is illustrated in the following examples.

Example: Solve the differential equation $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = x(t)$ given the input signal $x(t) = e^t u(t)$ and the initial conditions $y(0) = 1$, $\dot{y}(0) = 1$:

Since the poles of the system are -1 and -2 and the pole of the input signal is 1, we can guess that for all $t \geq 0$, $y(t)$ is a linear combination of e^{-t}, e^{-2t}, and e^t:

$$y(t) = c_1 e^{-t} + c_2 e^{-2t} + c_3 e^t. \tag{12}$$

Hence,

$$\begin{aligned} \dot{y}(t) &= -c_1 e^{-t} - 2c_2 e^{-2t} + c_3 e^t \\ \ddot{y}(t) &= c_1 e^{-t} + 4c_2 e^{-2t} + c_3 e^t. \end{aligned} \tag{13}$$

Using the initial conditions $y(0) = 1, \dot{y}(0) = 1$, we get

$$\begin{aligned} c_1 + c_2 + c_3 &= 1 \\ -c_1 - 2c_2 + c_3 &= 1. \end{aligned} \tag{14}$$

Also, using the equation $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = e^t$, we get

$$6c_3 = 1. \tag{15}$$

Hence, $c_1 = 2.5$, $c_2 = -\frac{5}{3}$, and $c_3 = \frac{1}{6}$. Therefore, $y(t) = 2.5e^{-t} - 1.6667e^{-2t} + 0.1667e^t$ for all $t \geq 0$.

Example: Solve the differential equation $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = x(t)$ given the input signal $x(t) = e^{-t}u(t)$ and the initial conditions $y(0) = 1$, $\dot{y}(0) = 1$:

Since the poles of the system are -1 and -2 and the pole of the input signal is also -1, we can guess that for all $t \geq 0$, $y(t)$ is a linear combination of e^{-t}, te^{-t}, and e^{-2t}:

$$y(t) = c_1 e^{-t} + c_2 t e^{-t} + c_3 e^{-2t}. \tag{16}$$

Hence,

$$\begin{aligned} \dot{y}(t) &= (c_2 - c_1)e^{-t} - c_2 t e^{-t} - 2c_3 e^{-2t} \\ \ddot{y}(t) &= (c_1 - 2c_2)e^{-t} + c_2 t e^{-t} + 4c_3 e^{-2t}. \end{aligned} \tag{17}$$

Using the initial conditions $y(0) = 1, \dot{y}(0) = 1$, we get

$$\begin{aligned} c_1 + c_3 &= 1 \\ c_2 - c_1 - 2c_3 &= 1. \end{aligned} \tag{18}$$

Also, using the equation $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = e^{-t}$, we get

$$c_2 = 1. \tag{19}$$

Hence, $c_1 = 2$, $c_2 = 1$, and $c_3 = -1$. Therefore, $y(t) = 2e^{-t} + te^{-t} - e^{-2t}$ for all $t \geq 0$.