One-Sided Laplace Transform and Differential Equations

As in the discrete-time case, the one-sided transform allows us to take initial conditions into account.

1 Preliminaries

- The one-sided Laplace transform of a continuous-time signal $x(t)$ is defined as $X(s) = \int_0^\infty x(t)e^{-st}dt$.

- If the one-sided Laplace transform of $x(t)$ is known to be $X(s)$, then the one-sided Laplace transform of $\frac{dx(t)}{dt}$ is $sX(s) - x(0)$.

Proof: By definition, the one-sided Laplace transform of $\frac{dx(t)}{dt}$ is

$$\int_0^\infty \frac{dx(t)}{dt}e^{-st}dt = x(t)e^{-st}|_0^\infty - \int_0^\infty x(t)(-se^{-st})dt$$

$$= -x(0) + sX(s) = sX(s) - x(0) \quad (1)$$

by using integration by parts.

- Generalizing, the one-sided Laplace transform of $\frac{d^2x(t)}{dt^2}$ is

$$s[sX(s) - x(0)] - \dot{x}(0) = s^2X(s) - sx(0) - \dot{x}(0). \quad (2)$$

This is obtained from the property above by recognizing that $\frac{d^2x(t)}{dt^2} = \frac{d}{dt} \left( \frac{dx(t)}{dt} \right)$. Similarly, the one-sided Laplace transforms of higher order derivatives can be written out in terms of $X(s)$. For instance, the one-sided Laplace transform of $\frac{d^3x(t)}{dt^3}$ is

$$s[sX(s) - x(0)] - \ddot{x}(0) = s^3X(s) - s^2x(0) - s\dot{x}(0) - \ddot{x}(0). \quad (3)$$

- In the case of the two-sided Laplace transform which was considered in the previous lecture notes, we saw that if the two-sided Laplace transform of $x(t)$ is known to be $X(s)$, then the two-sided Laplace transform of $\frac{dx(t)}{dt}$ is $sX(s)$. In the case of the one-sided Laplace transform, we get an additional term depending on the initial condition as seen above.
• By default, if we mention “Laplace transform” without explicitly specifying if it is one-sided or two-sided, then we implicitly mean a two-sided Laplace transform.

2 Solving Differential Equations

Given a differential equation, an input signal, and initial conditions, we have two methods to solve the differential equation:

1. Using the one-sided Laplace transform
2. Using the method of guessing exponentials.

These two methods are explained below with examples.

2.1 Solving Differential Equations Using The One-Sided Laplace Transform

In this method, we take the one-sided Laplace transform of both sides of the given differential equation. This procedure will result in an equation involving $Y(s)$, $X(s)$, and the given initial conditions. We then solve for $Y(s)$ in terms of $X(s)$ and the initial conditions. Then, taking the inverse Laplace transform of $Y(s)$ gives us $y(t)$. We will only consider differential equations wherein initial conditions at $t = 0$ are given and we are required to find $y(t)$ for $t \geq 0$, i.e., we interpret the differential equation as a causal system. This means that we must take the causal inverse Laplace transform of $Y(s)$ to find $y(t)$.

Example: Solve the differential equation $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = x(t)$ given the input signal $x(t) = e^t u(t)$ and the initial conditions $y(0) = 1$, $\dot{y}(0) = 1$:

Taking one-sided Laplace transforms of both sides, we have

$$s^2Y(s) - sy(0) - \dot{y}(0) + 3sY(s) - 3y(0) + 2Y(s) = X(s). \quad (4)$$

Hence,

$$Y(s) = \frac{X(s) + sy(0) + \dot{y}(0) + 3y(0)}{s^2 + 3s + 2} = \frac{1}{s-1} + s + 4$$

$$\frac{s^2 + 3s + 2}{s^2 + 3s + 2}$$

\[ \frac{1}{s-1} + s + 4 \]
\[
= \frac{1 + (s - 1)(s + 4)}{(s^2 + 3s + 2)(s - 1)} = \frac{s^2 + 3s - 3}{(s^2 + 3s + 2)(s - 1)} \tag{5}
\]

where we have used the fact that the one-sided Laplace transform of \(e^t u(t)\) is \(\frac{1}{s - 1}\). Decomposing into partial fractions, we get

\[
Y(s) = \frac{0.1667}{s - 1} + \frac{2.5}{s + 1} + \frac{-1.6667}{s + 2}. \tag{6}
\]

Hence, for all \(t \geq 0\),

\[
y(t) = 0.1667e^t + 2.5e^{-t} - 1.6667e^{-2t}. \tag{7}
\]

**Example:** Solve the differential equation \(\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = x(t)\) given the input signal \(x(t) = e^{-t}u(t)\) and the initial conditions \(y(0) = 1, \dot{y}(0) = 1\):

As in the previous example, taking one-sided Laplace transforms of both sides, we have

\[
s^2Y(s) - sy(0) - \dot{y}(0) + 3sY(s) - 3y(0) + 2Y(s) = X(s). \tag{8}
\]

Therefore,

\[
Y(s) = \frac{s^2Y(s) - sy(0) - \dot{y}(0) + 3sY(s) - 3y(0) + 2Y(s)}{s^2 + 3s + 2} = \frac{s^2Y(s) - sy(0) - \dot{y}(0) + 3sY(s) - 3y(0) + 2Y(s)}{s^2 + 3s + 2} = \frac{1 + (s + 1)(s + 4)}{(s^2 + 3s + 2)(s + 1)} \tag{9}
\]

where we have used the fact that the one-sided Laplace transform of \(e^{-t}u(t)\) is \(\frac{1}{s + 1}\). Decomposing into partial fractions, we get

\[
Y(s) = \frac{2}{s + 1} + \frac{1}{(s + 1)^2} + \frac{-1}{s + 2}. \tag{10}
\]

Hence, for all \(t \geq 0\),

\[
y(t) = 2e^{-t} + te^{-t} - e^{-2t}. \tag{11}
\]
2.2 Solving Differential Equations Using The Method of Guessing Exponentials

As discussed in the lecture notes on inverse Laplace transforms, we expect that \( y(t) \) will include terms of the form \( e^{p_i t} \) where \( p_1, \ldots, p_n \) are the poles of the system (i.e., the poles of \( H(s) \)) and the poles of the input signal (i.e., the poles of \( X(s) \)). If any of the poles is repeated (i.e., has multiplicity greater than one), then we get additional terms. In general, if a pole \( p_i \) is repeated \( k \) times, then \( y(t) \) includes the terms \( e^{p_i t} u(t), \ldots, t^{k-1} e^{p_i t} u(t) \). Hence, by finding the poles of \( H(s) \) and the poles of \( X(s) \), we can guess the form of \( y(t) \). To find the coefficients of the terms in the guessed form of \( y(t) \), we use the initial conditions and the given differential equation. The procedure is illustrated in the following examples.

**Example:** Solve the differential equation \( \ddot{y}(t) + 3\dot{y}(t) + 2y(t) = x(t) \) given the input signal \( x(t) = e^t u(t) \) and the initial conditions \( y(0) = 1, \dot{y}(0) = 1 \):

Since the poles of the system are \(-1\) and \(-2\) and the pole of the input signal is \(1\), we can guess that for all \( t \geq 0 \), \( y(t) \) is a linear combination of \( e^{-t} \), \( e^{-2t} \), and \( e^t \):

\[
y(t) = c_1 e^{-t} + c_2 e^{-2t} + c_3 e^t. \tag{12}
\]

Hence,

\[
\dot{y}(t) = -c_1 e^{-t} - 2c_2 e^{-2t} + c_3 e^t \\
\ddot{y}(t) = c_1 e^{-t} + 4c_2 e^{-2t} + c_3 e^t. \tag{13}
\]

Using the initial conditions \( y(0) = 1, \dot{y}(0) = 1 \), we get

\[
c_1 + c_2 + c_3 = 1 \\
-c_1 - 2c_2 + c_3 = 1. \tag{14}
\]

Also, using the equation \( \ddot{y}(t) + 3\dot{y}(t) + 2y(t) = e^t \), we get

\[
6c_3 = 1. \tag{15}
\]

Hence, \( c_1 = 2.5, c_2 = -\frac{5}{3}, \) and \( c_3 = \frac{1}{6} \). Therefore, \( y(t) = 2.5e^{-t} - 1.6667e^{-2t} + 0.1667e^t \) for all \( t \geq 0 \).

**Example:** Solve the differential equation \( \ddot{y}(t) + 3\dot{y}(t) + 2y(t) = x(t) \) given the input signal \( x(t) = e^{-t} u(t) \) and the initial conditions \( y(0) = 1, \dot{y}(0) = 1 \):
Since the poles of the system are $-1$ and $-2$ and the pole of the input signal is also $-1$, we can guess that for all $t \geq 0$, $y(t)$ is a linear combination of $e^{-t}$, $te^{-t}$, and $e^{-2t}$:

$$y(t) = c_1 e^{-t} + c_2 te^{-t} + c_3 e^{-2t}.$$ \hfill (16)

Hence,

$$\dot{y}(t) = (c_2 - c_1) e^{-t} - c_2 te^{-t} - 2c_3 e^{-2t}$$

$$\ddot{y}(t) = (c_1 - 2c_2) e^{-t} + c_2 te^{-t} + 4c_3 e^{-2t}.$$ \hfill (17)

Using the initial conditions $y(0) = 1$, $\dot{y}(0) = 1$, we get

$$c_1 + c_3 = 1$$

$$c_2 - c_1 - 2c_3 = 1.$$ \hfill (18)

Also, using the equation $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = e^{-t}$, we get

$$c_2 = 1.$$ \hfill (19)

Hence, $c_1 = 2$, $c_2 = 1$, and $c_3 = -1$. Therefore, $y(t) = 2e^{-t} + te^{-t} - e^{-2t}$ for all $t \geq 0$. 

5