Laplace Transform and Continuous-Time Frequency Response

1 Definition of Laplace Transform

- Given a continuous-time signal $x(t)$, the Laplace transform of $x(t)$ is defined as

$$X(s) \;=\; \int_{-\infty}^{\infty} x(t)e^{-st}dt. \tag{1}$$

  Note that $X(s)$ is a function which takes a complex number s and returns a complex number $X(s)$, i.e., $X(s)$ is a function which maps the complex plane into the complex plane. The set of values of s for which the integral in (1) is well-defined is called the *Region of Convergence (ROC)* of $X(s)$.

- We will see in Section 2 that the ROC is a region in the complex plane which is bounded by lines parallel to the imaginary axis (i.e., the line $\mathrm{Re}(s) = 0$).

- **Examples:**

  1. The Laplace transform of $\delta(t)$ is $\int_{-\infty}^{\infty} \delta(t)e^{-st}dt = 1$. The Laplace transform integral is well defined for all values of s. Hence, the ROC is the entire complex plane.
  2. The Laplace transform of $u(t)$ is $\int_{-\infty}^{\infty} u(t)e^{-st}dt = \int_{0}^{\infty} e^{-st}dt = \frac{-e^{-st}}{s}\Big|_0^{\infty} = \frac{1-e^{-s\infty}}{s}$ where by $e^{-s\infty}$, we mean the limit $\lim_{t\to\infty} e^{-st}$. If the real part of s is positive, then $e^{-s\infty} = 0$. If the real part of s is not positive, then $e^{-s\infty}$ is not well defined. Therefore, the Laplace transform of $u(t)$ is $\frac{1}{s}$ with the ROC $\mathrm{Re}(s) > 0$.
  3. Consider the signal $x(t) = e^{at}u(t)$. The Laplace transform of $x(t)$ is

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} e^{at}u(t)e^{-st}dt \;=\; \int_{0}^{\infty} e^{at}e^{-st}dt \\ &= \int_{0}^{\infty} e^{-(s-a)t}dt \;=\; \frac{1-e^{-(s-a)\infty}}{s-a}. \end{aligned} \tag{2}$$

As in the previous example, we note that the real part of $(s-a)$ should be greater than zero for $e^{-(s-a)\infty}$ to be well defined. The requirement that the real part of $(s-a)$ should be greater than zero is equivalent to the requirement that the real part of s should be greater than the real part of a, i.e., that $\text{Re}(s) > \text{Re}(a)$. Hence, the Laplace transform of $e^{at}u(t)$ is $\frac{1}{s-a}$ with the ROC $\text{Re}(s) > \text{Re}(a)$.

4. Consider the signal $x(t) = e^{-at}u(-t)$. The Laplace transform of $x(t)$ is

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} e^{-at}u(-t)e^{-st}dt = \int_{-\infty}^{0} e^{-at}e^{-st}dt \\ &= \int_{-\infty}^{0} e^{-(s+a)t}dt = \frac{-1+e^{(s+a)\infty}}{s+a}. \end{aligned} \quad (3)$$

Similar to the previous examples, we conclude that, since $e^{(s+a)\infty}$ is 0 if $\text{Re}(s+a) < 0$ and not well defined otherwise, the Laplace transform of $e^{-at}u(-t)$ is $\frac{-1}{s+a}$ with the ROC $\text{Re}(s) < -\text{Re}(a)$.

2 ROC of the Laplace Transform

- The observations on the ROC in the preceding examples can be generalized as follows. If the signal is right sided (in other words, if there is a time t_0 before which the signal is zero, i.e., $x(t) = 0$ for all $t < t_0$), then the ROC of the Laplace transform of the signal is to the right hand side of a line parallel to the imaginary axis. If the signal is left sided (in other words, if there is a time t_0 after which the signal is zero, i.e., $x(t) = 0$ for all $t > t_0$), then the ROC of the Laplace transform of the signal is to the left hand side of a line parallel to the imaginary axis. If the signal is two sided, then the ROC is the region between two lines parallel to the imaginary axis. In particular, if the signal is causal (which definitely means that the signal is right sided since a causal signal takes the value zero for all $t < 0$), then the ROC of the Laplace transform of the signal is to the right hand side of a line parallel to the imaginary axis. Similarly, if the signal is anticausal, then the ROC of the Laplace transform of the signal is to the left hand side of a line parallel to the imaginary axis.

- Given any signal $x(t)$, the ROC of its Laplace transform is bounded by a pole of $X(s)$ in the sense that the boundary of the ROC has a pole on it. If $x(t)$ is causal, then the ROC of its Laplace transform lies to the right hand side of all its poles and the boundary of the ROC is at its rightmost pole.
- A system is BIBO stable if and only if its impulse response satisfies the property $\int_{-\infty}^{\infty} |h(t)|dt < \infty$. This is equivalent to requiring that the ROC of the Laplace transform of $h(t)$ should include the imaginary axis. For a causal signal, we know that the ROC of its Laplace transform lies to the right hand side of all its poles with its boundary being at its rightmost pole. Hence, for a causal signal, BIBO stability is equivalent to requiring that all the poles should lie in the left half plane (i.e., the half of the complex s plane containing complex numbers with negative real parts).

> A causal continuous-time LTI system with transfer function $H(s)$ is BIBO stable if and only if all the poles of $H(s)$ lie in the left half plane.

3 Properties of the Laplace Transform

1. **Linearity of the Laplace transform:** If the Laplace transform of a signal $x(t)$ is $X(s)$, then the Laplace transform of $\alpha x(t)$ is $\alpha X(s)$ for any constant α. Also, if the Laplace transforms of two signals $x_1(t)$ and $x_2(t)$ are $X_1(s)$ and $X_2(s)$, then the Laplace transform of $\alpha x_1(t)+\beta x_2(t)$ is $\alpha X_1(s) + \beta X_2(s)$ for any constants α and β.
2. **Convolution in time domain is equivalent to multiplication in Laplace domain:** If the Laplace transforms of two signals $x_1(t)$ and $x_2(t)$ are $X_1(s)$ and $X_2(s)$, respectively, then the Laplace transform of the signal $x_1(t) * x_2(t)$ is $X_1(s)X_2(s)$.

   Proof: By definition, the Laplace transform of $x_1(t) * x_2(t)$ is

$$\int_{-\infty}^{\infty} [x_1(t) * x_2(t)]e^{-st}dt = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau\right] e^{-st}dt$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)e^{-s(t-\tau+\tau)}d\tau dt \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(\tau)e^{-s\tau}x_2(t-\tau)e^{-s(t-\tau)}d\tau dt \\
&= \int_{-\infty}^{\infty} x_1(\tau)e^{-s\tau}d\tau \int_{-\infty}^{\infty} x_2(\tau_1)e^{-s\tau_1}d\tau_1 \\
&= X_1(s)X_2(s) \qquad (4)
\end{aligned}
$$

where the dummy variable $\tau_1 = t - \tau$ was used.

3. **Time shift in time domain is equivalent to modulation in Laplace domain:** If the Laplace transform of $x(t)$ is $X(s)$, then the Laplace transform of $x(t-t_0)$ is $e^{-st_0}X(s)$.

   Proof: By definition, the Laplace transform of $x(t-t_0)$ is

$$
\begin{aligned}
\int_{-\infty}^{\infty} x(t-t_0)e^{-st}dt &= \int_{-\infty}^{\infty} x(t-t_0)e^{-s(t-t_0)}e^{-st_0}dt \\
&= e^{-st_0}\int_{-\infty}^{\infty} x(t_1)e^{-st_1}dt_1 = e^{-st_0}X(s) \qquad (5)
\end{aligned}
$$

   where t_1 is the dummy variable $t_1 = t - t_0$.

   Example: Since the Laplace transform of $\delta(t)$ is 1, the Laplace transform of $\delta(t-t_0)$ is e^{-st_0}.

4. **Modulation in time domain is equivalent to shift in Laplace domain:** If the Laplace transform of $x(t)$ is $X(s)$, then the Laplace transform of $e^{s_0t}x(t)$ is $X(s-s_0)$.

   Proof: By definition, the Laplace transform of $e^{s_0t}x(t)$ is

$$
\int_{-\infty}^{\infty} e^{s_0t}x(t)e^{-st}dt = \int_{-\infty}^{\infty} x(t)e^{-(s-s_0)t}dt = X(s-s_0). \qquad (6)
$$

   Example: Since the Laplace transform of $u(t)$ is $\frac{1}{s}$, the Laplace transform of $e^{at}u(t)$ is $\frac{1}{s-a}$.

5. If the Laplace transform of $x(t)$ is $X(s)$, then the Laplace transform of $x^*(t)$ is $X^*(s^*)$.

   Proof: By definition, the Laplace transform of $x^*(t)$ is

$$
\begin{aligned}
\int_{-\infty}^{\infty} x^*(t)e^{-st}dt &= \int_{-\infty}^{\infty} [x(t)e^{-s^*t}]^*dt \\
&= [\int_{-\infty}^{\infty} x(t)e^{-s^*t}dt]^* = X^*(s^*). \qquad (7)
\end{aligned}
$$

Example: The Laplace transform of the signal $x(t) = e^{j\omega t}u(t)$ can be found to be $X(s) = \frac{1}{s-j\omega}$. Therefore, the Laplace transform of $x^*(t) = e^{-j\omega t}u(t)$ is $X^*(s^*) = \frac{1}{(s^*-j\omega)^*} = \frac{1}{s+j\omega}$.

6. If the Laplace transform of $x(t)$ is $X(s)$, then the Laplace transform of $x(at)$ is $\frac{1}{|a|}X(\frac{s}{a})$. In particular, if the Laplace transform of $x(t)$ is $X(s)$, then the Laplace transform of $x(-t)$ is $X(-s)$.

   Proof: By definition, the Laplace transform of $x(at)$ is

$$\int_{-\infty}^{\infty} x(at)e^{-st}dt \quad = \quad \int_{-\infty}^{\infty} x(at)e^{-s\frac{at}{a}}dt. \tag{8}$$

   If $a > 0$, we have $\int_{-\infty}^{\infty} x(at)e^{-s\frac{at}{a}}dt = \frac{1}{a}\int_{-\infty}^{\infty} x(t_1)e^{-s\frac{t_1}{a}}dt_1$ and if $a < 0$, we have $\int_{-\infty}^{\infty} x(at)e^{-s\frac{at}{a}}dt = -\frac{1}{a}\int_{-\infty}^{\infty} x(t_1)e^{-s\frac{t_1}{a}}dt_1$ where t_1 is the dummy variable $t_1 = at$. Therefore,

$$\int_{-\infty}^{\infty} x(at)e^{-s\frac{at}{a}}dt \quad = \quad \frac{1}{|a|}\int_{-\infty}^{\infty} x(t_1)e^{-\frac{s}{a}t_1}dt_1 \quad = \quad \frac{1}{|a|}X(\frac{s}{a}). \tag{9}$$

   In particular, if a is taken to be -1, we get the result that the Laplace transform of $x(-t)$ is $X(-s)$.

   Example: Since the Laplace transform of $e^t u(t)$ is $\frac{1}{s-1}$, the Laplace transform of $e^{at}u(at)$ is $\frac{1}{|a|}\frac{1}{\frac{s}{a}-1} = \frac{\mathrm{sgn}(a)}{s-a}$ where $\mathrm{sgn}(a)$ is the sign (± 1) of a.

7. **Differentiation in time domain is equivalent to multiplication by s in Laplace domain:** If the Laplace transform of $x(t)$ is $X(s)$, then the Laplace transform of $\frac{dx(t)}{dt}$ is $sX(s)$.

   Proof: Let the Laplace transform of $\frac{dx(t)}{dt}$ be denoted by $X_d(s)$. Then,

$$X_d(s) \quad = \quad \int_{-\infty}^{\infty} \frac{dx(t)}{dt}e^{-st}dt. \tag{10}$$

   Using integration by parts, we have

$$X_d(s) \quad = \quad x(t)e^{-st}|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} x(t)[-se^{-st}]dt. \tag{11}$$

   Within the ROC, $x(t)e^{-st}$ is zero at both the limit as $t \to \infty$ and the limit as $t \to -\infty$ since, by definition, the integral $\int_{-\infty}^{\infty} x(t)e^{-st}dt$ is well defined in the ROC. Hence, (11) simplifies to $X_d(s) = sX(s)$.

Example: In previous examples, we found the Laplace transforms of $e^{j\omega t}u(t)$ and $e^{-j\omega t}u(t)$ to be $\frac{1}{s-j\omega}$ and $\frac{1}{s+j\omega}$, respectively. Therefore, the Laplace transform of the signal $x_1(t) = \sin(\omega t)u(t)$ is

$$\begin{aligned} X_1(s) &= \frac{1}{2j}\left[\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right] \\ &= \frac{\omega}{s^2+\omega^2}. \end{aligned} \tag{12}$$

Noting that $\frac{d\sin(\omega t)u(t)}{dt} = \omega\cos(\omega t)u(t) + \sin(\omega t)\delta(t) = \omega\cos(\omega t)u(t)$, we find that the Laplace transform of the signal $x_2(t) = \cos(\omega t)u(t)$ is

$$X_2(s) = \frac{1}{\omega}s\frac{\omega}{s^2+\omega^2} = \frac{s}{s^2+\omega^2}. \tag{13}$$

Note that

$$\begin{aligned} \frac{dx_2(t)}{dt} &= -\omega\sin(\omega t)u(t) + \cos(\omega t)\delta(t) \\ &= -\omega\sin(\omega t)u(t) + \delta(t) = -\omega x_1(t) + \delta(t). \end{aligned} \tag{14}$$

Hence, it should be true that

$$sX_2(s) = -\omega X_1(s) + 1. \tag{15}$$

This can indeed be easily verified to be true.

Example: The current-voltage relation of a capacitor is $i_C = C\frac{dv_C}{dt}$. Hence, $\frac{V_C(s)}{I_C(s)} = \frac{1}{sC}$. In analogy with the relation $v = iR$ for a resistor, we say that the impedance of a capacitor is $\frac{1}{sC}$. Similarly, the current-voltage relation of an inductor is $v_L = L\frac{di}{dt}$ implying that $\frac{V_L(s)}{I_L(s)} = sL$. Hence, the impedance of an inductor is sL.

8. **Integration in time domain is equivalent to multiplication by $\frac{1}{s}$ in Laplace domain:** If the Laplace transform of $x(t)$ is $X(s)$, then the Laplace transform of $\int_{-\infty}^{t} x(\tau)d\tau$ is $\frac{1}{s}X(s)$.

   Proof: From the previous property, we know that if the Laplace transform of a signal $x_1(t)$ is $X_1(s)$, then the Laplace transform of $\frac{dx_1(t)}{dt}$ is $sX_1(s)$. Defining $x_1(t) = \int_{-\infty}^{t} x(\tau)d\tau$, we have $x(t) = \frac{dx_1(t)}{dt}$. Therefore, $X(s) = sX_1(s)$ which means that $X_1(s) = \frac{1}{s}X(s)$.

Alternative proof: Consider a system with the impulse response $h(t) = u(t)$, i.e, with the transfer function $H(s) = \frac{1}{s}$. If $x(t)$ is the input to this system, the output is $y(t) = u(t) * x(t) = \int_{-\infty}^{t} x(\tau)d\tau$. Hence, the transfer function of $\int_{-\infty}^{t} x(\tau)d\tau$ is $H(s)X(s) = \frac{1}{s}X(s)$.

Example: The Laplace transform of $u(t)$ is $\frac{1}{s}$. Hence, the Laplace transform of $tu(t) = \int_{-\infty}^{t} u(\tau)d\tau$ is $\frac{1}{s^2}$. In general, by applying the same procedure $(n-1)$ times, we find that the Laplace transform of the signal $\frac{t^{n-1}}{(n-1)!}u(t)$ is $\frac{1}{s^n}$.

9. **Multiplication by $-t$ in time domain is equivalent to differentiation in Laplace domain:** If the Laplace transform of $x(t)$ is $X(s)$, then the Laplace transform of $-tx(t)$ is $\frac{dX(s)}{ds}$.

   Proof: By definition, the Laplace transform of $-tx(t)$ is

$$\begin{aligned}\int_{-\infty}^{\infty}[-tx(t)]e^{-st}dt &= \int_{-\infty}^{\infty} x(t)\frac{de^{-st}}{ds}dt \\ &= \frac{d}{ds}\int_{-\infty}^{\infty} x(t)e^{-st}dt = \frac{dX(s)}{ds}. \end{aligned} \tag{16}$$

   Example: The Laplace transform of $e^{-\alpha t}u(t)$ is $\frac{1}{s+\alpha}$. Hence, the Laplace transform of $-te^{-\alpha t}u(t)$ is $\frac{d}{ds}\frac{1}{s+\alpha} = -\frac{1}{(s+\alpha)^2}$, i.e, the Laplace transform of $te^{-\alpha t}u(t)$ is $\frac{1}{(s+\alpha)^2}$. In general for any positive integer n, the Laplace transform of $\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$ is $\frac{1}{(s+\alpha)^n}$.

4 Inverse Laplace Transform

Given a function $H(s)$ and the ROC, the inverse Laplace transform involves the problem of finding the signal $h(t)$ such that the Laplace transform of $h(t)$ is $H(s)$ with the given ROC. We will see three methods for finding the inverse Laplace transform:

1. Using partial fractions
2. Guessing an exponential solution
3. Using integration

These methods are explained below.

4.1 Inverse Laplace Transform Using Partial Fractions

In this method, we decompose the given function $H(s)$ into partial fractions and take the inverse Laplace transform of each term in the partial fraction. Remember that the ROC of a causal signal is the right hand side of a line parallel to the imaginary axis while the ROC of an anticausal signal is the left hand side of a line parallel to the imaginary axis. Hence, the inverse Laplace transform of $\frac{1}{s-a}$ given the ROC $\mathrm{Re}(s) > \mathrm{Re}(a)$ is $e^{at}u(t)$ while the inverse Laplace transform of $\frac{1}{s-a}$ given the ROC $\mathrm{Re}(s) < \mathrm{Re}(a)$ is $-e^{at}u(-t)$. The following examples will further illustrate the method of finding the inverse Laplace transform by using partial fractions.

Example: Find the inverse Laplace transform of $H(s) = \frac{1}{(s+1)(s+2)}$ with the ROC $\mathrm{Re}(s) > -1$: Taking partial fractions, we have

$$\frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}. \tag{17}$$

The poles of the two terms in the above equation are -1 and -2, respectively. The given ROC is to the right hand side of both the lines $\mathrm{Re}(s) = -1$ and $\mathrm{Re}(s) = -2$. Therefore, both terms in (17) yield causal terms. Hence,

$$h(t) = e^{-t}u(t) - e^{-2t}u(t). \tag{18}$$

Example: Find the inverse Laplace transform of $H(s) = \frac{1}{(s+1)(s+2)}$ with the ROC $-2 < \mathrm{Re}(s) < -1$: Taking partial fractions, we have

$$\frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}. \tag{19}$$

The poles of the two terms in the above equation are -1 and -2, respectively. The given ROC is to the right hand side of the line $\mathrm{Re}(s) = -2$ and to the left hand side of the line $\mathrm{Re}(s) = -1$. Therefore, the first term in (19) yields an anticausal term while the second term in (19) yields a causal term. Hence,

$$h(t) = -e^{-t}u(-t) - e^{-2t}u(t). \tag{20}$$

Example: Find the inverse Laplace transform of $H(s) = \frac{1}{(s+1)(s+2)^2}$ with the ROC $\mathrm{Re}(s) > -1$: Taking partial fractions, we obtain

$$\frac{1}{(s+1)(s+2)^2} = \frac{1}{s+1} + \frac{-1}{s+2} + \frac{-1}{(s+2)^2}. \tag{21}$$

As in the previous example, the given ROC implies that all the terms in (21) yield causal terms. Hence,

$$h(t) \quad = \quad e^{-t}u(t) - e^{-2t}u(t) - te^{-2t}u(t). \tag{22}$$

4.2 Inverse Laplace Transform by Guessing an Exponential Solution

From the above examples, we see that when we take partial fractions, we get terms involving each of the poles $p_1, \ldots, p_n$ of $H(s)$ so that the inverse Laplace transform involves terms of the form $e^{p_i t}u(t)$. If any of the poles are repeated, then the partial fraction expansion includes additional terms. In general, if a pole p_i is repeated k times, then the partial fraction expansion contains the terms $\frac{1}{(s-p_i)}, \ldots, \frac{1}{(s-p_i)^k}$. Hence, the inverse Laplace transform includes the terms $e^{p_i t}u(t), \ldots, t^{k-1}e^{p_i t}u(t)$. This means that we can *guess* the form of the inverse Laplace transform easily by just finding the poles of $H(s)$. However, the guessed form of the inverse Laplace transform involves unknown coefficients which need to be determined using the differential equation associated with the given $H(s)$.

Example: Find the inverse Laplace transform of $H(s) = \frac{1}{(s+1)(s+2)}$ with the ROC $\text{Re}(s) > -1$: The poles of $H(s)$ are -1 and -2. Because the ROC is to the right hand side of the lines parallel to the imaginary axis and passing through the poles, the signal $h(t)$ must be causal. Hence, we can guess the form of $h(t)$ to be

$$h(t) \quad = \quad c_1 e^{-t}u(t) + c_2 e^{-2t}u(t) \tag{23}$$

with c_1 and c_2 being coefficients to be determined. Considering a system with transfer function $H(s)$, we have $Y(s) = X(s)H(s)$ if x is the input to the system and y the output. Hence, $Y(s)[s^2 + 3s + 2] = X(s)$ and the differential equation associated with $H(s)$ is

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) \quad = \quad x(t). \tag{24}$$

By definition, $h(t)$ is the response of the system when $\delta(t)$ is applied as the input signal. Hence, $h(t)$ satisfies the equation

$$\ddot{h}(t) + 3\dot{h}(t) + 2h(t) \quad = \quad \delta(t). \tag{25}$$

We have guessed the form of $h(t)$ to be as in (23). Hence, $\dot{h}(t)$ and $\ddot{h}(t)$ are of the form

$$\begin{aligned} \dot{h}(t) &= -c_1 e^{-t}u(t) - 2c_2 e^{-2t}u(t) + (c_1 + c_2)\delta(t) \\ \ddot{h}(t) &= c_1 e^{-t}u(t) + 4c_2 e^{-2t}u(t) - (c_1 + 2c_2)\delta(t) + (c_1 + c_2)\dot{\delta}(t). \end{aligned} \tag{26}$$

Substituting the guessed forms of $h(t)$, $\dot{h}(t)$, and $\ddot{h}(t)$ into (25) and equating the coefficients of $\delta(t)$, $\dot{\delta}(t)$, $e^{-t}u(t)$, and $e^{-2t}u(t)$ on the two sides of the equation, we get the following relations between c_1 and c_2:

$$\begin{aligned} 2c_1 + c_2 &= 1 \\ c_1 + c_2 &= 0. \end{aligned} \tag{27}$$

Hence, $c_1 = 1$ and $c_2 = -1$. Therefore, from (23), $h(t) = e^{-t}u(t) - e^{-2t}u(t)$.

4.3 Inverse Laplace Transform by Using Integration

The inverse Laplace transform can be evaluated as

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}ds \tag{28}$$

where σ is any real constant such that the line $(\sigma - j\infty, \sigma + j\infty)$ lies in the ROC. Note that the line $(\sigma - j\infty, \sigma + j\infty)$ is parallel to the imaginary axis. (28) implies that if $X(s)$ is known on the line $(\sigma - j\infty, \sigma + j\infty)$, then $x(t)$ can be found which means that the values of $X(s)$ for all values of s can be found. In other words, all the information content in $X(s)$ is encapsulated within the values of $X(s)$ on the line $(\sigma - j\infty, \sigma + j\infty)$ in the sense that knowing $X(s)$ on the line $(\sigma - j\infty, \sigma + j\infty)$ is equivalent to knowing $X(s)$ throughout the complex plane.

5 Frequency Response

If the input signal is the sinusoidal signal $x(t) = Ae^{j\phi}e^{j\omega t}u(t)$, then

$$Y(s) = H(s)X(s) = H(s)Ae^{j\phi}\frac{1}{s - j\omega}. \tag{29}$$

$y(t)$ can be found by taking the (causal) inverse Laplace transform of $Y(s)$. This can be done, for instance, by partial fractions. If the system is BIBO stable, then the terms in the partial fraction expansion corresponding to poles of the system yield terms that exponentially go to zero as $t \to \infty$. Hence, for a BIBO stable system, it can be shown that the output signal resulting due to the input signal $x(t) = Ae^{j\phi}e^{j\omega t}u(t)$ converges at steady state to the scaled and shifted sinusoidal signal

$$y_s(t) = Ae^{j\phi}|H(j\omega)|e^{j\angle H(j\omega)}e^{j\omega t}u(t). \tag{30}$$

The same conclusion can also be reached using convolution. Assuming that the system is BIBO stable, we can neglect the homogeneous response (i.e., the effect of initial conditions). Hence,

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)Ae^{j\phi}e^{j\omega(t-\tau)}u(t-\tau)d\tau. \\ &= Ae^{j\phi}e^{j\omega t}\int_{-\infty}^{t} h(\tau)e^{-j\omega\tau}d\tau. \end{aligned} \tag{31}$$

At steady state, i.e., as $t \to \infty$, we get

$$\begin{aligned} y(t) &= Ae^{j\phi}e^{j\omega t}\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau \\ &= Ae^{j\phi}e^{j\omega t}H(j\omega) \;=\; Ae^{j\phi}|H(j\omega)|e^{j\angle H(j\omega)}e^{j\omega t}u(t). \end{aligned} \tag{32}$$

Similarly, for a BIBO stable system, the output signal resulting due to the input signal $x(t) = A\sin(\omega t + \phi)u(t)$ converges at steady state to the scaled and shifted sinusoidal signal

$$y_s(t) = A|H(j\omega)|\sin(\omega t + \phi + \angle H(j\omega))u(t). \tag{33}$$