Inverse z-Transforms and Difference Equations

1 Preliminaries

- We have seen that given any signal $x[n]$, the two-sided z-transform is given by $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$ and $X(z)$ *converges* in a region of the complex plane called the **region of convergence (ROC)**. The inverse z-transform addresses the reverse problem, i.e., to find $x[n]$ given $X(z)$ and the ROC.

- The inverse z-transform for the one-sided z-transform is also defined analogous to above, i.e., given a function $X(z)$ and a ROC, find the signal $x[n]$ whose one-sided z-transform is $X(z)$ and has the specified ROC. Since the one-sided z-transform involves, by definition, only the values of $x[n]$ for $n \geq 0$, the inverse one-sided z-transform is always a causal signal so that the ROC is always the exterior of the circle through the largest pole.

- There are four common ways of finding the inverse z-transform:

  - Using long division
  - Using partial fractions
  - Using contour integrals
  - Using the associated difference equation.

  The first three methods are explained below in Sections 2-4. The method of finding the inverse z-transform using the associated difference equation is explained in Section 6. Of these methods, the two which are easiest to apply are the method of partial fractions and the method of using the associated difference equation. Among these two methods, the method of partial fractions is easier to use if $X(z)$ does not have repeated poles and the method of using the associated difference equation is easier if $X(z)$ has repeated poles. Recall that the *poles* of $X(z)$ are the values of z for which $X(z)$ is ∞. For example, if $X(z)$ is given to be $\frac{1}{1-\alpha z^{-1}}$, then $X(z)$ has one pole at α. As another example, consider $X(z) = \frac{1}{(1-z^{-1})(1-2z^{-1})}$. This $X(z)$ has the two poles

1 and 2 and neither pole is repeated. On the other hand, the function $X(z) = \frac{1}{(1-z^{-1})^2}$ has one pole at 1 which is repeated twice (in other words, the pole has *multiplicity* two).

2 Finding inverse z-transform using long division

To apply this method, we try to express $X(z)$ as a Laurent series $X(z) = \sum_{n=-\infty}^{\infty} x_n z^{-n}$. Then, by comparing with the definition of the z-transform, we find that $x[n] = x_n$.

Example 2.1 Consider $X(z) = \frac{1}{1-z^{-1}}$ with ROC $1 < |z| < \infty$. To expand $X(z)$ as a Laurent series, we note that $|z^{-1}| < 1$ in the ROC. Hence,

$$\frac{1}{1-z^{-1}} = 1 + z^{-1} + z^{-2} + z^{-3} + \dots \tag{1}$$

which implies that $x[n] = 1$ for $n \geq 0$ and $x[n] = 0$ for $n < 0$, i.e., $x[n] = u[n]$. Note that the property $|z^{-1}| < 1$ is essential for the equation (1) to be valid.

Example 2.2 Consider $X(z) = \frac{1}{1-z^{-1}}$ with ROC $0 < |z| < 1$. We cannot use the expansion (1) since $|z^{-1}|$ is bigger than 1 in the ROC. To obtain a Laurent series expansion of $X(z)$, rewrite $X(z)$ as

$$X(z) = \frac{z}{z-1} = \frac{-z}{1-z}. \tag{2}$$

Since $|z| < 1$ in the ROC, we have the identity

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \dots \tag{3}$$

which is valid in the ROC. Hence,

$$\begin{aligned} X(z) &= -z(1 + z + z^2 + z^3 + \dots) \\ &= -(z + z^2 + z^3 + z^4 + \dots) \\ &= -\sum_{n=1}^{\infty} z^n. \end{aligned} \tag{4}$$

By comparing with the definition $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$, we conclude that $x[n] = -1$ for $n \leq -1$ and $x[n] = 0$ for $n > 0$, i.e., $x[n] = -u[-n-1]$.

In general, we have the following results (which we had derived earlier in the treatment of z-transforms) which are valid for all α:

If $X(z) = \frac{1}{1-\alpha z^{-1}}$ and the ROC is $|\alpha| < |z| < \infty$, then $x[n] = \alpha^n u[n]$.

If $X(z) = \frac{1}{1-\alpha z^{-1}}$ and the ROC is $0 < |z| < |\alpha|$, then $x[n] = -\alpha^n u[-n-1]$.

3 Finding inverse z-transform using partial fractions

We have seen above how to find the inverse z-transform of a function $X(z)$ of the form $\frac{1}{1-\alpha z^{-1}}$. Given any function $X(z)$, the method of partial fractions attempts to write $X(z)$ as a combination of terms of the form $\frac{1}{(1-\alpha_i z^{-1})^k}, i = 1, \ldots, N, k = 1, \ldots, N_i$ where the poles of $X(z)$ are $\alpha_1, \ldots, \alpha_N$ with the multiplicity of pole α_i being N_i.

The simplest case is when the poles of $X(z)$ are all *simple*, i.e., no pole is repeated. In this case, all poles have multiplicity 1 and $X(z)$ is of the form

$$X(z) = \frac{P(z)}{(1-\alpha_1 z^{-1})(1-\alpha_2 z^{-1})\ldots(1-\alpha_N z^{-1})} \quad (5)$$

with $P(z)$ being some function (which does not vanish at any α_i). The function $X(z)$ can be decomposed into partial fractions as

$$X(z) = \sum_{i=1}^{N} \frac{A_i}{1-\alpha_i z^{-1}} \quad (6)$$

with A_i being constants.

If any pole is repeated more than once in $X(z)$, then the partial fraction expansion contains more terms. For instance, if the pole α_i is repeated N_i

times, then the partial fraction expansion of $X(z)$ will contain the terms $\frac{A_{i1}}{(1-\alpha_i z^{-1})}$, $\frac{A_{i2}}{(1-\alpha_i z^{-1})^2}$, ..., $\frac{A_{iN_i}}{(1-\alpha_i z^{-1})^{N_i}}$. Hence, in general, if the poles of $X(z)$ are $\alpha_1, \ldots, \alpha_N$ with the multiplicity of pole α_i being N_i, then the partial fraction expansion of $X(z)$ is

$$X(z) = \sum_{i=1}^{N} \sum_{k=1}^{N_i} \frac{A_{ik}}{(1-\alpha_i z^{-1})^k}. \tag{7}$$

The examples below illustrate the procedure to find the partial fraction expansion of a given function $X(z)$ and to use the partial fraction expansion to find the inverse z-transform of $X(z)$ given a ROC.

Example 3.1 Consider $X(z) = \frac{1}{(1-z^{-1})(1-2z^{-1})}$ given the ROC $1 < |z| < 2$. This $X(z)$ has two poles at 1 and 2. The multiplicity of each pole is 1. Hence, the partial fraction expansion of $X(z)$ is of the form

$$\frac{1}{(1-z^{-1})(1-2z^{-1})} = \frac{A_1}{1-z^{-1}} + \frac{A_2}{1-2z^{-1}}. \tag{8}$$

To find A_1 and A_2, multiply both sides of (8) by $(1-z^{-1})(1-2z^{-1})$. We obtain

$$1 = A_1(1-2z^{-1}) + A_2(1-z^{-1}). \tag{9}$$

Equating the constant terms and the coefficients of z^{-1}, we get the two equations

$$\begin{aligned} A_1 + A_2 &= 1 \\ -2A_1 - A_2 &= 0. \end{aligned} \tag{10}$$

This is a set of two equations in the two unknowns A_1 and A_2. Solving, we get

$$\begin{aligned} A_1 &= -1 \\ A_2 &= 2. \end{aligned} \tag{11}$$

Hence, the partial fraction expansion of $X(z)$ is

$$\frac{1}{(1-z^{-1})(1-2z^{-1})} = \frac{-1}{1-z^{-1}} + \frac{2}{1-2z^{-1}}. \tag{12}$$

The ROC is given to be $1 < |z| < 2$. This ROC is in the exterior of the circle with center at the origin and radius 1 and is in the interior of the circle with center at the origin and radius 2. Hence, the term in the inverse z-transform resulting from the first term in the right hand side of (12) will be a causal term while the term resulting from the second term in the right hand side of (12) will be an anti-causal term. Therefore, we obtain the inverse z-transform

$$\begin{aligned} x[n] &= (-1)(1)^n u[n] + (2)(-(2)^n u[-n-1]) \\ &= -u[n] - 2^{n+1} u[-n-1]. \end{aligned} \tag{13}$$

Example 3.2 Find the inverse z-transform of $X(z) = \frac{1}{(1-z^{-1})(1-2z^{-1})}$ given the ROC $0 < |z| < 1$. The function $X(z)$ is the same as in the example 3.1. Hence, the partial fraction expansion (12) is valid. Since, in this example, the ROC is given to be $0 < |z| < 1$, the ROC is in the interior of the circle with center at the origin and radius 1 and also in the interior of the circle with center at the origin and radius 2. Hence, both terms in (12) give anti-causal terms in the inverse z-transform. Therefore, the inverse z-transform is

$$\begin{aligned} x[n] &= (-1)(-(1)^n u[-n-1]) + (2)(-(2)^n u[-n-1]) \\ &= (1 - 2^{n+1}) u[-n-1]. \end{aligned} \tag{14}$$

Example 3.3 Find the inverse z-transform of $X(z) = \frac{1}{(1-z^{-1})(1-2z^{-1})}$ given the ROC $2 < |z| < \infty$. The function $X(z)$ is the same as in the examples 3.1 and 3.2. The difference is that the ROC is given to be $2 < |z| < \infty$ which is in the exterior of the circle with center at the origin and radius 1 and also in the exterior of the circle with center at the origin and radius 2. Hence, both terms in (12) give causal terms in the inverse z-transform. Therefore, the inverse z-transform is

$$x[n] = (-1)(1)^n u[n] + (2)(2)^n u[n] = (-1 + 2^{n+1}) u[n]. \tag{15}$$

Example 3.4 Consider $X(z) = \frac{1+z^{-1}}{(1-z^{-1})(1-2z^{-1})(1-3z^{-1})}$ given the ROC $2 < |z| < 3$. The poles of $X(z)$ are at 1, 2, and 3 and the multiplicity of each pole is 1. Hence, the partial fraction expansion of $X(z)$ is of the form

$$X(z) = \frac{A_1}{1-z^{-1}} + \frac{A_2}{1-2z^{-1}} + \frac{A_3}{1-3z^{-1}}. \tag{16}$$

To find A_1, A_2, and A_3, we multiply both sides by $(1-z^{-1})(1-2z^{-1})(1-3z^{-1})$ and equate the constant terms, the coefficients of z^{-1}, and the coefficients of z^{-2} on the two sides of the resulting equation. This procedure yields the equations

$$\begin{aligned} A_1 + A_2 + A_3 &= 1 \\ -5A_1 - 4A_2 - 3A_3 &= 1 \\ 6A_1 + 3A_2 + 2A_3 &= 0. \end{aligned} \tag{17}$$

Solving, $A_1 = 1$, $A_2 = -6$, and $A_3 = 6$. Hence, the partial fraction expansion of $X(z)$ is

$$X(z) = \frac{1}{1-z^{-1}} - \frac{6}{1-2z^{-1}} + \frac{6}{1-3z^{-1}}. \tag{18}$$

Since the given ROC is $2 < |z| < 3$, we infer that the first and second terms in (18) provide causal terms in the inverse z-transform while the third term provides an anti-causal term, i.e., the inverse z-transform of $X(z)$ given the ROC $2 < |z| < 3$ is

$$x[n] = (1)^n u[n] - 6(2)^n u[n] + 6(-(3)^n u[-n-1]). \tag{19}$$

Example 3.5 Consider $X(z) = \frac{1-4z^{-1}}{(1-z^{-1})^2(1-2z^{-1})}$ given the ROC $2 < |z| < \infty$. The poles of $X(z)$ are 1 and 2. The pole at 2 is a simple pole while the pole at 1 has multiplicity two. Hence, using the general expression (7), the partial fraction expansion of $X(z)$ is of the form

$$X(z) = \frac{A_{11}}{1-z^{-1}} + \frac{A_{12}}{(1-z^{-1})^2} + \frac{A_{21}}{1-2z^{-1}}. \tag{20}$$

The constants A_{11}, A_{12}, and A_{21} can be found using essentially the same technique as above. Multiplying both sides of (20) by $(1-z^{-1})^2(1-2z^{-1})$, we obtain

$$\begin{aligned} 1-4z^{-1} &= A_{11}(1-z^{-1})(1-2z^{-1}) + A_{12}(1-2z^{-1}) + A_{21}(1-z^{-1})^2 \\ &= A_{11}(1-3z^{-1}+2z^{-2}) + A_{12}(1-2z^{-1}) \\ &\quad + A_{21}(1-2z^{-1}+z^{-2}). \end{aligned} \tag{21}$$

Equating the constant term, the coefficients of z^{-1}, and the coefficients of z^{-2}, we obtain

$$\begin{aligned} A_{11} + A_{12} + A_{21} &= 1 \\ -3A_{11} - 2A_{12} - 2A_{21} &= -4 \\ 2A_{11} + A_{21} &= 0. \end{aligned} \tag{22}$$

Solving the above set of equations, we find $A_{11} = 2$, $A_{12} = 3$, and $A_{21} = -4$. Hence, the partial fraction expansion of $X(z)$ is

$$X(z) = \frac{2}{1 - z^{-1}} + \frac{3}{(1 - z^{-1})^2} - \frac{4}{1 - 2z^{-1}}. \tag{23}$$

From the given ROC $2 < |z| < \infty$, we infer that all the terms (23) yield causal terms in the inverse z-transform. We know the causal inverse z-transform of $\frac{1}{1-\alpha z^{-1}}$ is $\alpha^n u[n]$. To find the inverse z-transform of $\frac{1}{(1-z^{-1})^2}$, we can use the property of z-transforms that if the z-transform of $x[n]$ is $X(z)$, then the z-transform of $nx[n]$ is $-z\frac{dX(z)}{dz}$. This implies that the z-transform of $nu[n]$ is $\frac{z^{-1}}{(1-z^{-1})^2}$. Noting that

$$\begin{aligned} \frac{1}{(1 - z^{-1})^2} &= \frac{1 - z^{-1} + z^{-1}}{(1 - z^{-1})^2} \\ &= \frac{1}{1 - z^{-1}} + \frac{z^{-1}}{(1 - z^{-1})^2}, \end{aligned} \tag{24}$$

the (causal) inverse z-transform of $\frac{1}{(1-z^{-1})^2}$ is $u[n] + nu[n]$.

Hence, from (23), the inverse z-transform of $X(z)$ given the ROC $2 < |z| < \infty$ is

$$x[n] = 2u[n] + 3(n + 1)u[n] - 4(2)^n u[n]. \tag{25}$$

4 Finding inverse z-transform using contour integrals

The Cauchy integral theorem says that

$$\frac{1}{2\pi j} \oint_\Gamma z^{k-1} dz = \begin{cases} 1 \text{ if } k = 0 \\ 0 \text{ if } k \neq 0 \end{cases} \tag{26}$$

where Γ is any contour (path) around the origin. This result can be used to find the inverse z-transform given $X(z)$ and a ROC. Let Γ be any contour around the origin and lying in the ROC.

By the definition of the z-transform, we have $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$. Multiplying both sides by z^{k-1} and taking the contour integral of both sides, we obtain

$$\begin{aligned} \oint_\Gamma X(z)z^{k-1}dz &= \oint_\Gamma \sum_{n=-\infty}^{\infty} x[n]z^{-n+k-1}dz \\ &= \sum_{n=-\infty}^{\infty} x[n] \oint_\Gamma z^{-n+k-1}dz. \end{aligned} \tag{27}$$

By the Cauchy integral theorem $\oint_\Gamma z^{-n+k-1}dz$ is $2\pi j$ if $n = k$ and 0 for all other values of n. Hence,

$$\oint_\Gamma X(z)z^{k-1}dz = 2\pi j x[k] \tag{28}$$

so that

$$x[k] = \frac{1}{2\pi j} \oint_\Gamma X(z)z^{k-1}dz. \tag{29}$$

The equation (29) gives the inverse z-transform of $X(z)$. The contour Γ should be chosen such that Γ lies in the ROC.

The direct application of (29) is usually not straightforward since it involves the evaluation of a contour integral. In this course, we will not use the method of contour integrals to actually compute inverse z-transforms. However, the relation (29) is of conceptual interest since it implies that to find the inverse z-transform, we only need to know $X(z)$ on the contour Γ, i.e., we do not need to know $X(z)$ for other values of z. This means that $X(z)$ at all points in the complex plane can be inferred by simply knowing its values on the contour Γ. Hence, there is a great deal of redundancy in the definition of $X(z)$. This is one of the mathematical motivations for the introduction of the discrete-time Fourier transform.

5 Difference Equations

A difference equation is any equation of the form

$$y[n] = \sum_{l=1}^{N} a_l y[n-l] + \sum_{k=0}^{M} b_k x[n-k] \qquad (30)$$

with M and N being positive integers and $a_1, \ldots, a_N, b_0, \ldots, b_M$ being constant coefficients. The general problem of solving difference equations can be stated as follows:

Given the difference equation (30), an input signal $x[n]$, and initial conditions $y[-1], \ldots, y[-N]$, find the output signal $y[n]$ for all $n \geq 0$.

The most basic way of solving this problem is by manually iterating the equation (30) as

$$\begin{aligned} y[0] &= \sum_{l=1}^{N} a_l y[-l] + \sum_{k=0}^{M} b_k x[-k] \\ y[1] &= \sum_{l=1}^{N} a_l y[1-l] + \sum_{k=0}^{M} b_k x[1-k] \\ y[2] &= \sum_{l=1}^{N} a_l y[2-l] + \sum_{k=0}^{M} b_k x[2-k] \\ &\vdots \end{aligned} \qquad (31)$$

Of course, this approach is useful only for small values of n. If we wanted to find $y[1000]$, it would require 1000 iterations ... a lot of work!

There are two general and efficient methods of solving difference equations:

- Using one-sided z-transforms,
- By guessing exponential solutions.

5.1 Solving difference equations using one-sided z-transforms

Recall that the one-sided z-transform of $x[n]$ is defined as $X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$. If the one-sided z-transform of $x[n]$ is $X(z)$, then the one-sided z-transform of $x[n-n_0]$ can be found as follows: Let the one-sided z-transform of $x[n-n_0]$ be denoted by $X_{n_0}(z)$. Then,

$$\begin{aligned} X_{n_0}(z) &= \sum_{n=0}^{\infty} x[n-n_0]z^{-n} \\ &= \sum_{n=0}^{\infty} x[n-n_0]z^{-(n-n_0)}z^{-n_0} \\ &= z^{-n_0} \sum_{n_1=-n_0}^{\infty} x[n_1]z^{-n_1} \end{aligned} \tag{32}$$

where $n_1 = n - n_0$. Hence,

$$\begin{aligned} X_{n_0}(z) &= z^{-n_0}\Big[\sum_{n_1=0}^{\infty} x[n_1]z^{-n_1} + \sum_{n_1=-n_0}^{-1} x[n_1]z^{-n_1}\Big] \\ &= z^{-n_0}\Big[X(z) + \sum_{n_1=-n_0}^{-1} x[n_1]z^{-n_1}\Big]. \end{aligned} \tag{33}$$

Note that unlike the case of two-sided z-transforms, we get an extra term $z^{-n_0}\sum_{n_1=-n_0}^{-1} x[n_1]z^{-n_1}$ in the one-sided z-transform of $x[n-n_0]$.

Taking the one-sided z-transform of both sides of (30),

$$\begin{aligned} Y(z) &= \sum_{l=1}^{N} a_l z^{-l}(Y(z) + \sum_{n_1=-l}^{-1} y[n_1]z^{-n_1}) \\ &\quad + \sum_{k=0}^{M} b_k z^{-k}(X(z) + \sum_{n_1=-k}^{-1} x[n_1]z^{-n_1}) \end{aligned} \tag{34}$$

where $X(z)$ and $Y(z)$ are the one-sided z-transforms of $x[n]$ and $y[n]$, respectively. Simplifying,

$$\begin{aligned} [1 - \sum_{l=1}^{N} a_l z^{-l}]Y(z) &= \sum_{k=0}^{M} b_k z^{-k} X(z) + \sum_{k=0}^{M} b_k z^{-k} \sum_{n_1=-k}^{-1} x[n_1]z^{-n_1} \\ &\quad + \sum_{l=1}^{N} a_l z^{-l} \sum_{n_1=-l}^{-1} y[n_1]z^{-n_1}. \end{aligned} \tag{35}$$

Hence,

$$Y(z) = \underbrace{\frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{l=1}^{N} a_l z^{-l}} X(z) + \frac{\sum_{k=0}^{M} b_k z^{-k} \sum_{n_1=-k}^{-1} x[n_1]z^{-n_1}}{1 - \sum_{l=1}^{N} a_l z^{-l}}}_{\text{Forced response}}$$

$$+\underbrace{\frac{\sum_{l=1}^{N} a_l z^{-l} \sum_{n_1=-l}^{-1} y[n_1] z^{-n_1}}{1-\sum_{l=1}^{N} a_l z^{-l}}}_{\text{Homogeneous response}}. \tag{36}$$

The expression for $Y(z)$ in (36) consists of two parts. The first part depends only on the input signal and is called the *forced response*. The second part depends only on the initial conditions $y[-1], \ldots, y[-N]$ and is called the *homogeneous response*. This highlights one property of the linearity of the system. The total response is equal to the sum of the response due to the input signal and the response due to the initial conditions.

If the initial conditions are zero and if $x[n] = 0$ for $n < 0$, then we obtain

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1-\sum_{l=1}^{N} a_l z^{-l}}. \tag{37}$$

$H(z)$ is the z-transform of the impulse response and is known as the transfer function of the system. The values of z for which $H(z) = 0$ are known as the *zeros* of the system and the values of z for which $H(z) = \infty$ are known as the *poles* of the system. Note that each pole of $H(z)$ is also a pole of $Y(z)$. However, in general, $Y(z)$ also has poles introduced by $X(z)$.

Example 5.1 Solve the difference equation

$$y[n] = 2y[n-1] + x[n] \tag{38}$$

given the initial condition $y[-1] = 2$ and the input signal $x[n] = \delta[n] + u[n]$: Taking the one-sided z-transform of both sides of (38), we obtain

$$Y(z) = 2z^{-1}(Y(z) + y[-1]z) + X(z). \tag{39}$$

Hence,

$$Y(z) = \frac{2y[-1] + X(z)}{1-2z^{-1}} = \frac{4 + X(z)}{1-2z^{-1}}. \tag{40}$$

For the given input signal, we have $X(z) = 1 + \frac{1}{1-z^{-1}} = \frac{2-z^{-1}}{1-z^{-1}}$. Hence,

$$Y(z) = \frac{4}{1-2z^{-1}} + \frac{2-z^{-1}}{(1-z^{-1})(1-2z^{-1})}. \tag{41}$$

Using partial fractions,

$$\frac{2 - z^{-1}}{(1 - z^{-1})(1 - 2z^{-1})} \;=\; \frac{-1}{1 - z^{-1}} + \frac{3}{1 - 2z^{-1}}. \tag{42}$$

Noting that the original difference equation (38) is causal, we need to take the causal inverse z-transform of (41) to find $y[n]$. Hence,

$$y[n] \;=\; 4(2)^n u[n] + (-1)(1)^n u[n] + (3)(2)^n u[n]. \tag{43}$$

5.2 Solving difference equations by guessing exponential solutions

We know that the causal inverse z-transform of $\frac{1}{1-\alpha z^{-1}}$ is $\alpha^n u[n]$, i.e., an exponential. Hence, we can find the inverse z-transform of the function $Y(z)$ shown in (36) by finding out which exponentials will be contained in the inverse z-transform and *guessing* the solution to be a linear combination of the exponentials. We have also seen that if the partial fraction expansion of $Y(z)$ has a term of the form $\frac{1}{(1-\alpha z^{-1})^2}$, then $y[n]$ has terms of the forms $\alpha^n u[n]$ and $n\alpha^n u[n]$. In general, if α_i is a pole of $Y(z)$ with multiplicity N_i, then $y[n]$ will contain terms of the forms $\alpha^n u[n]$, $n\alpha^n u[n]$, $n^{N_i-1}\alpha^n u[n]$. Once we guess the solution $y[n]$ to be a linear combination of known terms, then we only need to find the coefficients. This can be done by generating enough equations through numerically iterating the original difference equation (30). This procedure is illustrated by the example below.

Example 5.2 Solve the difference equation

$$y[n] \;=\; -y[n-1] + 2y[n-2] + x[n] \tag{44}$$

given the initial conditions $y[-1] = 1$ and $y[-2] = 2$ and the input signal $x[n] = u[n]$:

The transfer function of the system (44) is

$$H(z) \;=\; \frac{1}{1 + z^{-1} - 2z^{-2}}. \tag{45}$$

The poles of $H(z)$ are the roots of the quadratic equation $1 + z^{-1} - 2z^{-2} = 0$. Hence, the poles of $H(z)$ are 1 and -2.

The z-transform of the input signal is $\frac{1}{1-z^{-1}}$ which has one pole at 1. Hence, the poles of $Y(z)$ are 1 and -2 with the multiplicity of the pole 1 being two and the multiplicity of the pole -2 being one. Therefore, we expect $y[n]$ to include terms of the forms $(1)^n u[n]$, $n(1)^n u[n]$, and $(-2)^n u[n]$, i.e.,

$$y[n] = c_1 u[n] + c_2 n u[n] + c_3 (-2)^n u[n]. \tag{46}$$

To find the three unknown coefficients c_1, c_2, and c_3, we need to generate three equations by numerically iterating (44) for $n = 0, 1$, and 2.

$$\begin{aligned} y[0] &= -y[-1] + 2y[-2] + x[0] = -1 + 2(2) + 1 = 4 \\ y[1] &= -y[0] + 2y[-1] + x[1] = -4 + 2(1) + 1 = -1 \\ y[2] &= -y[1] + 2y[0] + x[2] = -(-1) + 2(4) + 1 = 10. \end{aligned} \tag{47}$$

This yields the three equations

$$\begin{aligned} c_1 + c_2(0) + c_3(-2)^0 &= 4 \\ c_1 + c_2(1) + c_3(-2)^1 &= -1 \\ c_1 + c_2(2) + c_3(-2)^2 &= 10 \end{aligned} \tag{48}$$

i.e.,

$$\begin{aligned} c_1 + c_3 &= 4 \\ c_1 + c_2 - 2c_3 &= -1 \\ c_1 + 2c_2 + 4c_3 &= 10. \end{aligned} \tag{49}$$

Solving, $c_1 = \frac{20}{9}$, $c_2 = \frac{1}{3}$, and $c_3 = \frac{16}{9}$. Hence,

$$y[n] = \frac{20}{9} u[n] + \frac{1}{3} n u[n] + \frac{16}{9} (-2)^n u[n]. \tag{50}$$

6 Finding inverse z-transform using the associated difference equation

Given a function $H(z)$, finding the causal inverse z-transform $h[n]$ is equivalent to solving an associated difference equation with initial conditions zero. To see this, consider $H(z)$ to be of the form

$$H(z) = \frac{\sum_{k=0}^{M} v_k z^{-k}}{\sum_{l=0}^{N} q_l z^{-l}}. \tag{51}$$

Then, a difference equation whose transfer function is $H(z)$ can be found as follows:

$$Y(z) \;=\; X(z)H(z) \;=\; X(z)\frac{\sum_{k=0}^{M} v_k z^{-k}}{\sum_{l=0}^{N} q_l z^{-l}}. \tag{52}$$

Hence

$$Y(z)\sum_{l=0}^{N} q_l z^{-l} \;=\; X(z)\sum_{k=0}^{M} v_k z^{-k} \tag{53}$$

so that with zero initial conditions $y[-1], \ldots, y[-N]$ and zero $x[-1], \ldots, x[-M]$, we have

$$\sum_{l=0}^{N} q_l y[n-l] \;=\; \sum_{k=0}^{M} v_k x[n-k], \tag{54}$$

i.e.,

$$y[n] \;=\; -\sum_{l=1}^{N} q_l y[n-l] + \sum_{k=0}^{M} v_k x[n-k]. \tag{55}$$

The output signal $y[n]$ of (55) when the input signal is $x[n] = \delta[n]$ is, by definition, $h[n]$. Hence, the original problem of finding the causal inverse z-transform of $H(z)$ is equivalent to finding the impulse response of (55). This can be done using the method of guessing exponentials as explained in Section 5.2.