z-Transforms

1 Definitions

- Given an input signal $x[n]$, the two-sided z-transform is given by

$$X(z) = \sum_{k=-\infty}^{\infty} x[n]z^{-n}. \tag{1}$$

- Given an input signal $x[n]$, the one-sided z-transform is given by

$$X(z) = \sum_{k=0}^{\infty} x[n]z^{-n}. \tag{2}$$

- By default, the term z-transform means the two-sided z-transform, i.e., if we simply say z-transform without explicitly specifying if it is a one-sided z-transform or a two-sided z-transform, then we mean a two-sided z-transform.

- A series of the form (1) is called a *Laurent* series.

- **Notation:** The two-sided z-transform of a signal $x[n]$ is denoted by $z\{x[n]\}$.

2 Basic Properties of the z-transform

- If $x[n]$ is a causal signal, then the two-sided z-transform of $x[n]$ is equal to the one-sided z-transform of $x[n]$, i.e., if $x[n]$ is a causal signal, then the z-transform of $x[n]$ contains only negative powers of z.

- If $x[n]$ is of finite length and is zero for $n > N_2$ and $n < -N_1$, then the limits of summation in (1) reduce to $-N_1$ and N_2, i.e., the two-sided z-transform of such a signal $x[n]$ is

$$X(z) = \sum_{k=-N_1}^{N_2} x[n]z^{-n}. \tag{3}$$

- **Linearity:** $z\{\alpha x_1[n] + \beta x_2[n]\} = \alpha z\{x_1[n]\} + \beta z\{x_2[n]\}$
- $z\{x[n-n_0]\} = z^{-n_0} z\{x[n]\}$ ... time shift by n_0 is equivalent to multiplication of the z-transform by z^{-n_0}.

3 Examples

- $z\{\delta[n]\} = 1$
- $z\{\delta[n-n_0]\} = z^{-n_0}$ for any constant n_0.
- $z\{\alpha^n u[n]\} = \frac{1}{1-\alpha z^{-1}}$ for any constant α.
- $z\{-\alpha^n u[-n-1]\} = \frac{1}{1-\alpha z^{-1}}$ for any constant α.

4 More Properties of the z-transform

- $z\{x_1[n] * x_2[n]\} = z\{x_1[n]\} z\{x_2[n]\}$ ... convolution in the time domain is equivalent to multiplication in the z domain, i.e., convolution is mathematically equivalent to polynomial multiplication.

  Proof: Let $x[n] = x_1[n] * x_2[n]$, i.e., $x[n] = \sum_{k=-\infty}^{\infty} x_1[k] x_2[n-k]$. Then,

$$\begin{aligned} z\{x[n]\} &= \sum_{n=-\infty}^{\infty} x[n] z^{-n} \\ &= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1[k] x_2[n-k] z^{-n} \\ &= \sum_{k=-\infty}^{\infty} x_1[k] \sum_{n=-\infty}^{\infty} x_2[n-k] z^{-n} \\ &= \sum_{k=-\infty}^{\infty} x_1[k] z^{-k} \sum_{n=-\infty}^{\infty} x_2[n-k] z^{-(n-k)} \\ &= \sum_{k=-\infty}^{\infty} x_1[k] z^{-k} \sum_{n_1=-\infty}^{\infty} x_2[n_1] z^{-n_1} \qquad (4) \end{aligned}$$

where $n_1 = n - k$. Hence,

$$\begin{aligned} z\{x[n]\} &= \left(\sum_{k=-\infty}^{\infty} x_1[k]z^{-k}\right)\left(\sum_{n_1=-\infty}^{\infty} x_2[n_1]z^{-n_1}\right) \\ &= z\{x_1[n]\}z\{x_2[n]\}. \end{aligned} \tag{5}$$

- If $z\{x[n]\} = X(z)$, then $z\{a^n x[n]\} = X(a^{-1}z)$.

  Proof:

$$\begin{aligned} z\{a^n x[n]\} &= \sum_{n=-\infty}^{\infty} a^n x[n]z^{-n} \\ &= \sum_{n=-\infty}^{\infty} x[n](a^{-1}z)^{-n} \\ &= X(a^{-1}z). \end{aligned} \tag{6}$$

- If $z\{x[n]\} = X(z)$, then $z\{nx[n]\} = -z\frac{dX(z)}{dz}$.

  Proof: Note that

$$\begin{aligned} \frac{dX(z)}{dz} &= \frac{d}{dz}\sum_{n=-\infty}^{\infty} x[n]z^{-n} \\ &= \sum_{n=-\infty}^{\infty} (-n)x[n]z^{-n-1} \\ &= -z^{-1}\sum_{n=-\infty}^{\infty} nx[n]z^{-n}. \end{aligned} \tag{7}$$

  Hence,

$$z\{nx[n]\} = \sum_{n=-\infty}^{\infty} nx[n]z^{-n} = -z\frac{dX(z)}{dz}. \tag{8}$$

- If $x[n]$ is a causal signal, then $x[0] = \lim_{z\to\infty} X(z)$.

  Proof: If $x[n]$ is a causal signal, then $x[n] = 0$ for $n < 0$ so that $X(z)$ contains only negative powers of z, i.e., $X(z)$ is of the form

$$X(z) = x[0] + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3} + \ldots \tag{9}$$

  Taking $z \to \infty$, only the first term in (9) remains so that

$$\lim_{z\to\infty} X(z) = x[0]. \tag{10}$$

- If $z\{x[n]\} = X(z)$, then $z\{x[-n]\} = X(z^{-1})$.

  Proof:

$$\begin{aligned} z\{x[-n]\} &= \sum_{n=-\infty}^{\infty} x[-n]z^{-n} \\ &= \sum_{n_1=\infty}^{-\infty} x[n_1]z^{n_1} \end{aligned} \tag{11}$$

  where $n_1 = -n$. Hence,

$$\begin{aligned} z\{x[-n]\} &= \sum_{n_1=-\infty}^{\infty} x[n_1](z^{-1})^{-n_1} \\ &= X(z^{-1}). \end{aligned} \tag{12}$$

- If $z\{x[n]\} = X(z)$, then $z\{x^*[n]\} = X^*(z^*)$.

  Proof: Note that

$$\begin{aligned} X^*(z^*) &= \Big(\sum_{n=-\infty}^{\infty} x[n](z^*)^{-n}\Big)^* \\ &= \sum_{n=-\infty}^{\infty} x^*[n]z^{-n} \end{aligned} \tag{13}$$

  since $(z^*)^* = z$. Hence,

$$z\{x^*[n]\} = \sum_{n=-\infty}^{\infty} x^*[n]z^{-n} = X^*(z^*). \tag{14}$$

- If $z\{x[n]\} = X(z)$, then $z\{\mathrm{Re}(x[n])\} = \frac{1}{2}[X(z) + X^*(z^*)]$.

  Proof: From the property proved above, we know that $z\{x^*[n]\} = X^*(z^*)$. Noting that

$$\mathrm{Re}(x[n]) = \frac{1}{2}(x[n] + x^*[n]), \tag{15}$$

  the result follows.

- If $z\{x[n]\} = X(z)$, then $z\{\mathrm{Im}(x[n])\} = \frac{1}{2j}[X(z) - X^*(z^*)]$.

  Proof: We know that $z\{x^*[n]\} = X^*(z^*)$. Noting that

$$\mathrm{Im}(x[n]) = \frac{1}{2j}(x[n] - x^*[n]), \tag{16}$$

  the result follows.

5 Region of Convergence

- The set of values of z for which the summation in the definition of the z-transform (1) converges is called the Region of Convergence (ROC). The ROC is a subset of the complex plane.

- The ROC of the z-transform of a finite length signal is the entire complex plane. Since the z-transform of a finite length signal has only a finite number of terms in the summation, the z-transform converges for all z.

- From the theory of complex analysis, it is known that the ROC of the z-transform of any signal is a connected set and is bounded by circles (with the centers of the circles being at the origin of the complex plane). Hence, there are three possibilities for the ROC as shown in Figure 1:

  - Interior of a circle
  - Exterior of a circle
  - Annular region between two circles

- If $z = re^{j\omega}$, then $X(z) = \sum_{n=-\infty}^{\infty} x[n]r^{-n}e^{-j\omega n}$. It can be shown that $X(z)$ converges if and only if

$$\sum_{n=-\infty}^{\infty} |x[n]r^{-n}| \quad < \quad \infty. \tag{17}$$

  Example: If $x[n] = u[n]$, then (17) reduces to $\sum_{n=0}^{\infty} |r^{-n}| < \infty$ which holds if $|r^{-1}| < 1$, i.e., if $|r| > 1$. Hence, the ROC of the z-transform of $u[n]$ is $1 < |z| < \infty$, i.e., the exterior of a circle with radius 1.

  Example: If $x[n] = \alpha^n u[n]$, then (17) reduces to $\sum_{n=0}^{\infty} |\alpha^n r^{-n}| < \infty$ which holds if $|\alpha r^{-1}| < 1$, i.e., if $|r| > |\alpha|$. Hence, the ROC of the z-transform of $\alpha^n u[n]$ is $|\alpha| < |z| < \infty$, i.e., the exterior of a circle with radius $|\alpha|$.

  Example: If $x[n] = -\alpha^n u[-n-1]$, then (17) reduces to $\sum_{n=-1}^{-\infty} |\alpha^n r^{-n}| < \infty$ which holds if $|\alpha^{-1} r| < 1$, i.e., if $|r| < |\alpha|$. Hence, the ROC of the z-transform of $-\alpha^n u[-n-1]$ is $0 \leq |z| < |\alpha|$, i.e., the interior of a circle with radius $|\alpha|$.

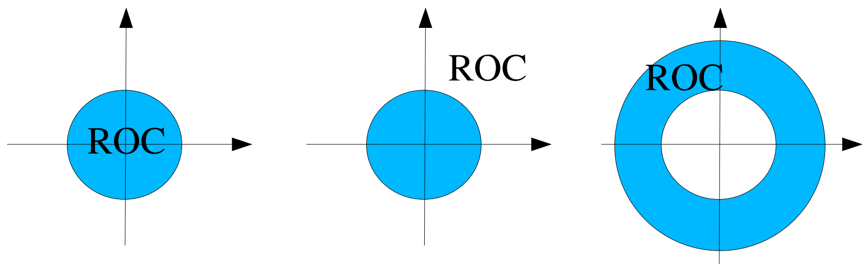

Figure 1: Possibilities for the region of convergence: (a) interior of a circle; (b) exterior of a circle; (c) annular region between two circles.

- From the above examples, it can be generalized that the ROC of the z-transform of a causal signal is the exterior of a circle and the ROC of the z-transform of an anti-causal signal (i.e., a signal which is zero for $n > 0$) is the interior of a circle.

- The above statement can be generalized to right-sided and left-sided signals. A right-sided signal is a signal which is zero for all n smaller than some constant n_1. A left-sided signal is a signal which is zero for all n greater than some constant n_1. The ROC of the z-transform of a right-sided signal is the exterior of a circle while the ROC of the z-transform of a left-sided signal is the interior of a circle (possibly not including the origin).

- **Example:** Consider the signal

$$x[n] = \begin{cases} \alpha^n \text{ for } n \geq 0 \\ \beta^n \text{ for } n \leq -1. \end{cases} \tag{18}$$

The signal $x[n]$ in (18) is a *two-sided* signal. If $|\alpha| < |\beta|$, then the ROC of the z-transform of the signal $x[n]$ is the annular region $|\alpha| < |z| < |\beta|$. If $|\alpha| \geq |\beta|$, then the z-transform of $x[n]$ does not converge for any z.

- **Special Case:** If $X(z)$ is a rational function (i.e., a ratio of polynomials), then the ROC is bounded by the poles of $X(z)$, i.e., the values of z for which $X(z)$ is infinity.

  Example: For the signal in (18), the z-transform is

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} \alpha^n z^{-n} + \sum_{n=-1}^{-\infty} \beta^n z^{-n} \\ &= \sum_{n=0}^{\infty} \alpha^n z^{-n} + \sum_{n=1}^{\infty} \beta^{-n} z^{n} \\ &= \frac{1}{1-\alpha z^{-1}} + \frac{1}{1-\beta^{-1}z} - 1 \\ &= \frac{1}{1-\alpha z^{-1}} + \frac{\beta^{-1}z}{1-\beta^{-1}z} \\ &= \frac{1}{1-\alpha z^{-1}} - \frac{1}{1-\beta z^{-1}}. \end{aligned} \tag{19}$$

  From (19), the poles of $X(z)$ are $z = \alpha$ and $z = \beta$. As seen above, if $|\alpha| < |\beta|$ the ROC of $X(z)$ is the annular region $|\alpha| < |z| < |\beta|$.

  Example: If $X(z)$ is a rational function, then the ROC of $X(z)$ is always bounded by poles. This means that the boundaries of the ROC have poles on them. It does *not* mean that if $X(z)$ has two poles, then the ROC is the annular region between the poles. For example, consider the signal $x[n] = \alpha^n u[n] + \beta^n u[n]$. It can be shown that the ROC of the z-transform of $X(z)$ is $\max(|\alpha|, |\beta|) < |z| < \infty$, i.e., the exterior of a circle with radius $\max(|\alpha|, |\beta|)$. The boundary of this ROC is the circle with radius $\max(|\alpha|, |\beta|)$ which has a pole on it (the pole being the larger of α and β). Note that the poles of $X(z)$ are α and β.