Convolution, FIR Systems, and IIR Systems

1 Some Definitions

- A signal which is 1 for $n = 0$ and 0 everywhere else is defined to be a discrete-time unit impulse and is denoted by $\delta[n]$, i.e.,

$$\delta[n] \;=\; \begin{cases} 1 \text{ for } n = 0 \\ 0 \text{ for } n \neq 0. \end{cases} \tag{1}$$

  The discrete-time unit impulse is also referred to as a *Kronecker delta* function.

- The response of a discrete-time system to a discrete-time unit impulse is said to be the impulse response of the system. The impulse response is usually denoted as $h[n]$.

- A signal which is 1 for $n \geq 0$ and 0 for $n < 0$ is said to be a discrete-time unit step function and is denoted by $u[n]$, i.e.,

$$u[n] \;=\; \begin{cases} 1 \text{ for } n \geq 0 \\ 0 \text{ for } n < 0. \end{cases} \tag{2}$$

- The *support* of a signal is defined to be the set of time instants at which the value of the signal is non-zero.
  Example: The support of $\delta[n]$ is the set $\{0\}$. The support of $u[n]$ is the set $\{0, 1, 2, 3, \ldots\}$.

- If the impulse response of a system is of finite length (i.e., if the support of the impulse response is a set with only a finite number of values), then the system is said to be a *Finite Impulse Response* (FIR) system (or FIR filter). Equivalently, an FIR system is a system in which the value of the output signal at any time depends only on the values of the input signal at a finite number of time instants.

- Any LTI FIR filter is of the general form

$$y[n] \;=\; \sum_{k=-M_1}^{M_2} b_k x[n-k] \tag{3}$$

with $b_{-M_1}, \ldots, b_{M_2}$ being the *coefficients* of the filter. M_1 and M_2 are positive constants. $L \triangleq M_1 + M_2 + 1$ is called the *length* of the filter.

Example: The moving average system given by

$$y[n] = \frac{x[n-1] + x[n] + x[n+1]}{3} \tag{4}$$

is an LTI FIR filter with $M_1 = 1$, $M_2 = 1$, $b_0 = b_{-1} = b_1 = \frac{1}{3}$, and $L = 3$.

- Any causal LTI FIR filter is of the general form

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] \tag{5}$$

with $b_0, \ldots, b_M$ being the *coefficients* of the filter. M is a positive constant and is called the *order* of the filter. $L \triangleq M + 1$ is called the *length* of the filter.

- The impulse response of the LTI FIR filter shown in (3) is given by

$$h[n] = \sum_{k=-M_1}^{M_2} b_k \delta[n-k] = b_n. \tag{6}$$

Hence, the coefficients of an LTI FIR filter are the values of the impulse response signal.

- A signal which is 0 for $n < 0$ is said to be a *causal signal*.
Exercise: Show that if the impulse response of an LTI system is a causal signal, then the system is causal.

2 Convolution

- If $x[n]$ is applied as the input to the LTI FIR filter shown in (3), then the output is

$$\begin{aligned} y[n] &= \sum_{k=-M_1}^{M_2} b_k x[n-k] \\ &= \sum_{k=-M_1}^{M_2} h[k] x[n-k] \triangleq h[n] * x[n]. \end{aligned} \tag{7}$$

- The *convolution* of signals x and h is defined to be the signal

$$y[n] \quad = \quad \sum_{k=-\infty}^{\infty} h[k]x[n-k] \tag{8}$$

  and is denoted by $h[n] * x[n]$. If the signal $h[n]$ has finite support, then the limits for k in the summation (8) can be replaced with the maximum and minimum values for which $h[k]$ is non-zero, thus obtaining (7).

- Proof that $\sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$: Any signal $x[n]$ can be decomposed into a sum of shifted impulses as

$$x[n] \quad = \quad \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]. \tag{9}$$

  Using time invariance, the output of a system in response to a shifted impulse is an appropriately shifted impulse response, i.e., if the input signal is $\delta[n-k]$, then the output signal is $h[n-k]$. Using linearity, the output of the system in response to a sum of signals is the sum of the corresponding outputs. Hence, using (9), the output of the system is

$$y[n] \quad = \quad \sum_{k=-\infty}^{\infty} x[k]h[n-k]. \tag{10}$$

  However, by (8), we know that $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$. Hence, $\sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$.

- Given the impulse response $h[n]$ of a system, the output signal corresponding to any given input signal $x[n]$ can be computed as $y[n] = h[n] * x[n]$. Hence, the impulse response completely characterizes the system.

If the impulse response of a system is $h[n]$ and the input signal is $x[n]$, then the output signal is $y[n] = x[n] * h[n] = h[n] * x[n]$.

3 Properties of Convolution

- **Commutative:** $x_1[n] * x_2[n] = x_2[n] * x_1[n]$
- **Associative:** $x_1[n] * (x_2[n] * x_3[n]) = (x_1[n] * x_2[n]) * x_3[n]$
- **Distributive:** $x_1[n] * (x_2[n] + x_3[n]) = x_1[n] * x_2[n] + x_1[n] * x_3[n]$
- The commutative, associative, and distributive properties of convolution can be used to do block diagram manipulation of systems. For instance, if two systems with impulse responses $h_1[n]$ and $h_2[n]$, respectively, are connected in cascade, then from an input-output perspective, the cascade is equivalent to an overall system with impulse response $h_1[n] * h_2[n]$. Since $h_1[n] * h_2[n] = h_2[n] * h_1[n]$, the order of systems in the cascade combination can be changed without affecting the overall system. Similarly, a parallel combination of two systems with impulse responses $h_1[n]$ and $h_2[n]$, respectively, is equivalent from an input-output perspective to an overall system with impulse response $h_1[n] + h_2[n]$.
- Convolution is mathematically equivalent to polynomial multiplication in the following sense: Define $X_1(z) = \sum_{n=-\infty}^{\infty} x_1[n]z^{-n}$ and $X_2(z) = \sum_{n=-\infty}^{\infty} x_2[n]z^{-n}$. Then, the value of the convolution $x_1[n]*x_2[n]$ at any time n_0 is the coefficient of the term z^{-n_0} in the product $X_1(z)X_2(z)$.

4 The Unit Step

- Recall that the unit step $u[n]$ is defined to be the signal which is 1 for $n \geq 0$ and 0 for $n < 0$, i.e.,

$$u[n] = \begin{cases} 1 \text{ for } n \geq 0 \\ 0 \text{ for } n < 0. \end{cases} \tag{11}$$

- From the definitions of the unit impulse and the unit step, we have $u[n] = \sum_{k=0}^{\infty} \delta[n-k]$. Hence, the step response (i.e., the response to a unit step) of an LTI system is given by $\sum_{k=0}^{\infty} h[n-k]$. If the system is causal, then $h[n] = 0$ for $n < 0$. Hence, the step response can be simplified to $\sum_{k=0}^{n} h[n-k]$. In other words, the value of the step response of an LTI causal system at time n is the sum of the impulse response values over the time interval $[0, n]$.

- The above discussion gives a recipe to compute the step response of an LTI system if we are given the impulse response of the system. On the other hand, if we are given the step response of an LTI system, we can find the impulse response by using the identities:

$$\delta[n] = u[n] - u[n-1] \tag{12}$$
$$\Longrightarrow h[n] = T\{u[n]\} - T\{u[n-1]\}. \tag{13}$$

5 IIR Systems

- The system which outputs a unit step in response to a δ function is given by

$$y[n] = y[n-1] + x[n] \tag{14}$$

with the initial condition $y[-1] = 0$, i.e., if $x[n] = \delta[n]$ is applied as the input signal to the system (14) initialized with $y[-1] = 0$, then the output signal is $y[n] = u[n]$. The system (14) is not an FIR system since the impulse response (which is the unit step in this case) is not of finite length. Such a system is called an Infinite Impulse Response (IIR) system.

- A general LTI IIR system is of the form

$$y[n] = \sum_{l=1}^{N} a_l y[n-l] + \sum_{k=0}^{M} b_k x[n-k] \tag{15}$$

with M and N being positive integers and $a_1, a_2, \ldots, a_N, b_0, b_1, \ldots, b_M$ being constant coefficients. An equation of the form (15) is called a *difference equation*.

5.1 Examples of Difference Equations

1. $y[n] = 0.5y[n-1] + x[n]$: The impulse response of this system is

$$h[n] = \frac{1}{2^n} u[n]. \tag{16}$$

Note that the impulse response goes to zero as $n \to \infty$.

2. $y[n] = 2y[n-1] + x[n]$: The impulse response of this system is

$$h[n] = 2^n u[n]. \tag{17}$$

Note that the impulse response goes to ∞ as $n \to \infty$.

3. $y[n] = y[n-1] + x[n]$: The impulse response of this system is

$$h[n] = u[n]. \tag{18}$$

Note that the impulse response stays constant as $n \to \infty$.

- **BIBO Stability**: A system is said to be Bounded Input Bounded Output (BIBO) stable if any bounded input signal produces a bounded output signal, i.e., a system is said to be BIBO stable if the following is true:
If a positive constant M_x exists such that

$$|x[n]| \leq M_x \text{ for all } n, \tag{19}$$

then a positive constant M_y exists such that

$$|y[n]| \leq M_y \text{ for all } n. \tag{20}$$

The condition for BIBO stability can be derived as follows. If $x[n]$ satisfies the bound (19), then

$$\begin{aligned} |y[n]| &= |\sum_{k=-\infty}^{\infty} h[k]x[n-k]| \\ &\leq \sum_{k=-\infty}^{\infty} |h[k]x[n-k]| \\ &\leq \sum_{k=-\infty}^{\infty} |h[k]|M_x \\ &= M_x \sum_{k=-\infty}^{\infty} |h[k]|. \end{aligned} \tag{21}$$

Hence, if $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$, then the bound (20) is satisfied with

$$M_y = M_x \sum_{k=-\infty}^{\infty} |h[k]|. \tag{22}$$

An LTI system with impulse response $h[n]$ is BIBO stable
if and only if $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$.

The condition that $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$ is equivalent to saying that $h[n]$ must be *absolutely summable*.

- All FIR systems are BIBO stable because only a finite number of $h[n]$ values are non-zero so that $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$ is definitely satisfied.

- An IIR system may or may not be BIBO stable. For instance, the second and third examples of difference equations considered above are not BIBO stable. For the second example, we found that the impulse response goes to infinity as $n \to \infty$, i.e., a bounded input (in this case, a unit impulse) results in an unbounded output. In the third example, if the input signal is the unit step, then the output signal (with the initial condition $y[-1] = 0$) can be computed as follows:

$$\begin{aligned} y[0] &= y[-1] + x[0] = 0 + 1 = 1 \\ y[1] &= y[0] + x[1] = 1 + 1 = 2 \\ y[2] &= y[1] + x[2] = 2 + 1 = 3 \\ &\vdots \end{aligned} \tag{23}$$

  In general, $y[n] = n + 1$. Hence, a bounded input (in this case, a unit step) results in an unbounded output. Thus, both the second and third examples considered above are not BIBO stable.

- Necessary and sufficient condition for the system

$$y[n] = ay[n-1] + x[n] \tag{24}$$

  to be BIBO stable is $|a| < 1$.