Lecture 4

Uncertain Interest Rates and Compounding

Overview

- Solving for unknown number of years
- Solving for unknown interest rate:
  - (Review of interpolation)
- Compounding more often than annually

Solving for an Unknown Number of Years

When the Number of Years n Is Unknown

- **Some problems require solving for a number of periods (NPER) given the other parameters**

- **We can always solve for n in closed form…**

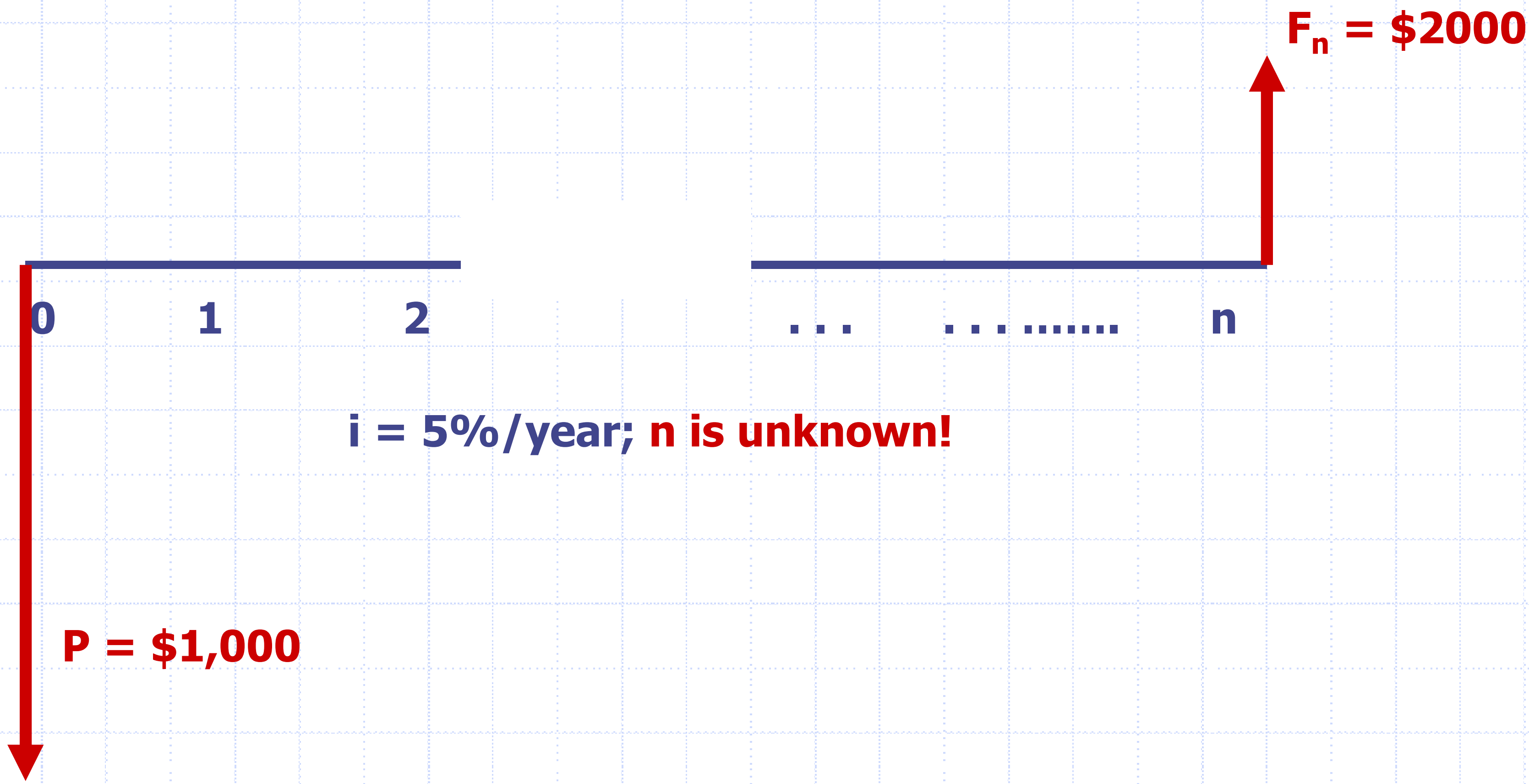
$$P = F\left[\frac{1}{(1+i)^n}\right]$$

$$A = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right]$$

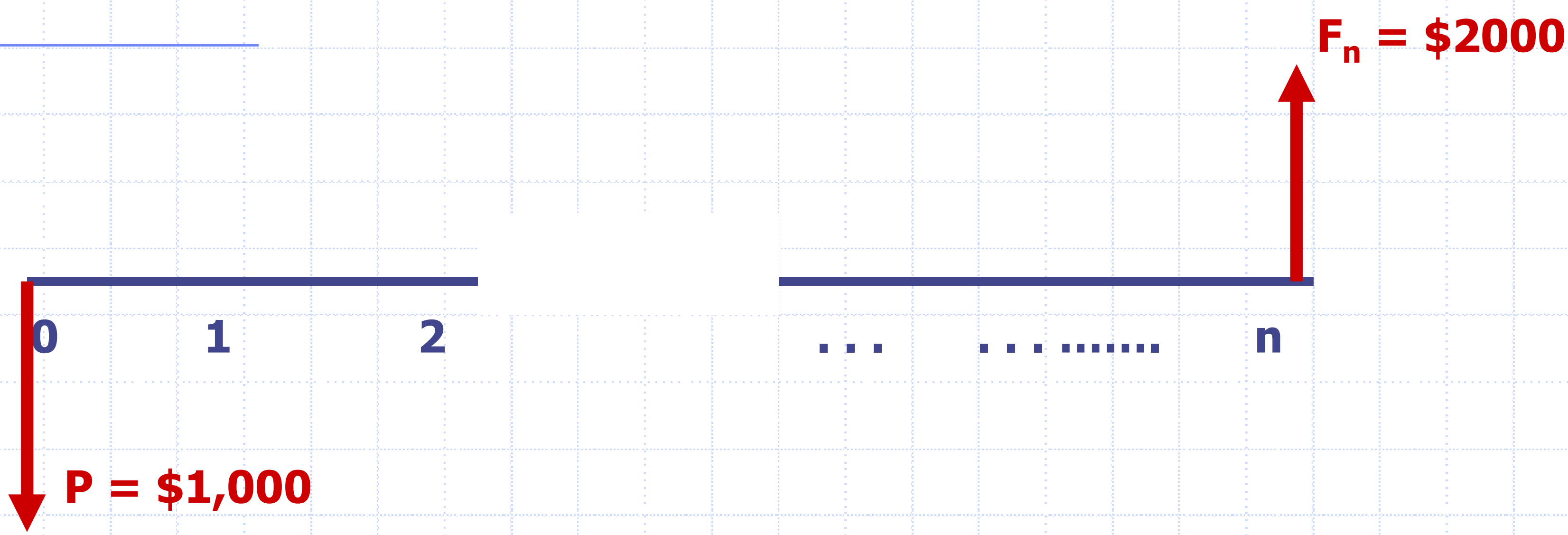
$$A = F\left[\frac{i}{(1+i)^n - 1}\right]$$

Example 2.14 (from the book)

- How long will it take for $1,000 to double in value, at an interest rate of 5%?



Example 2.14 (continued)



- $F_n = 1000\ (F/P, 5\%, n)$
- $2000 = 1000\ (1.05)^n$
- We can solve for n in closed form…

Example 2.14 (continued)

- Solving, we get…
- $(1.05)^n = 2000/1000$
- $n \ln(1.05) = \ln(2)$
- $n = \ln(2)/\ln(1.05)$
- $n = 0.693147/0.04879 =$ <u>14.2067 years</u>
- With compounding every year:
  - It will take **15 years** to amass $2,000
  - (Actually, a little more than $2,000!)

Example of the NPER Function

- **In Excel, one can do this using NPER:**

NPER(i, A, P, F)

	B	C
18	Blank: Ex 2.14 NPER	
19		
20	Present Value	-$1,000.00
21	Future Value	$2,000.00
22	A amount	$0.00
23	Interest rate	5.00%
24	Number of Periods	14.20669908

=NPER(C23,C22,C20,C21)

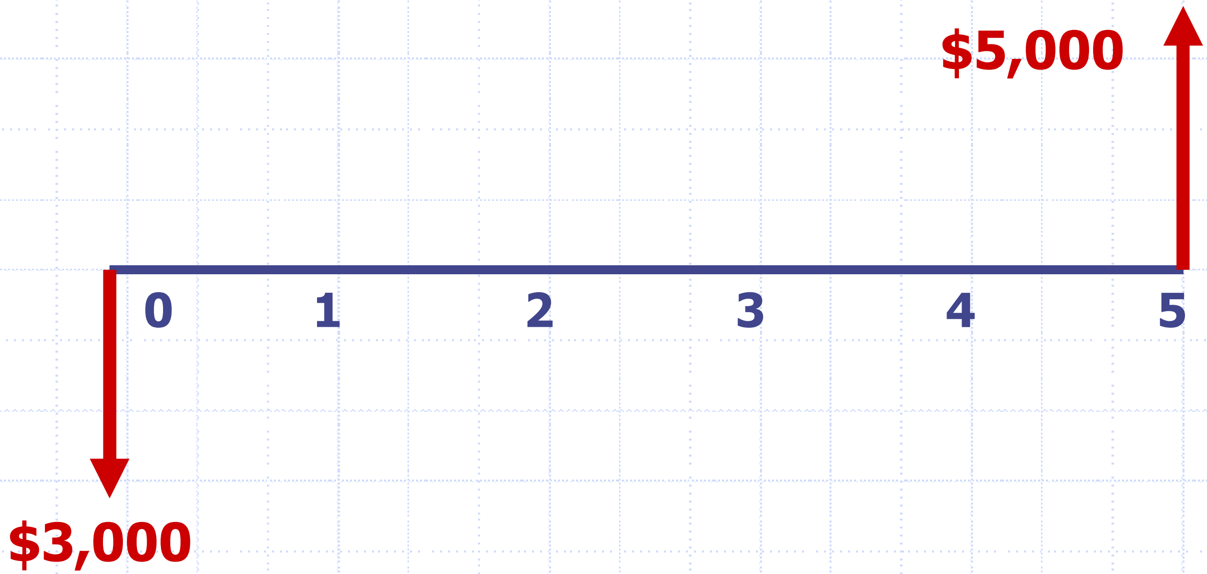
Solving for an Unknown Interest Rate

When the Interest Rate i Is Unknown

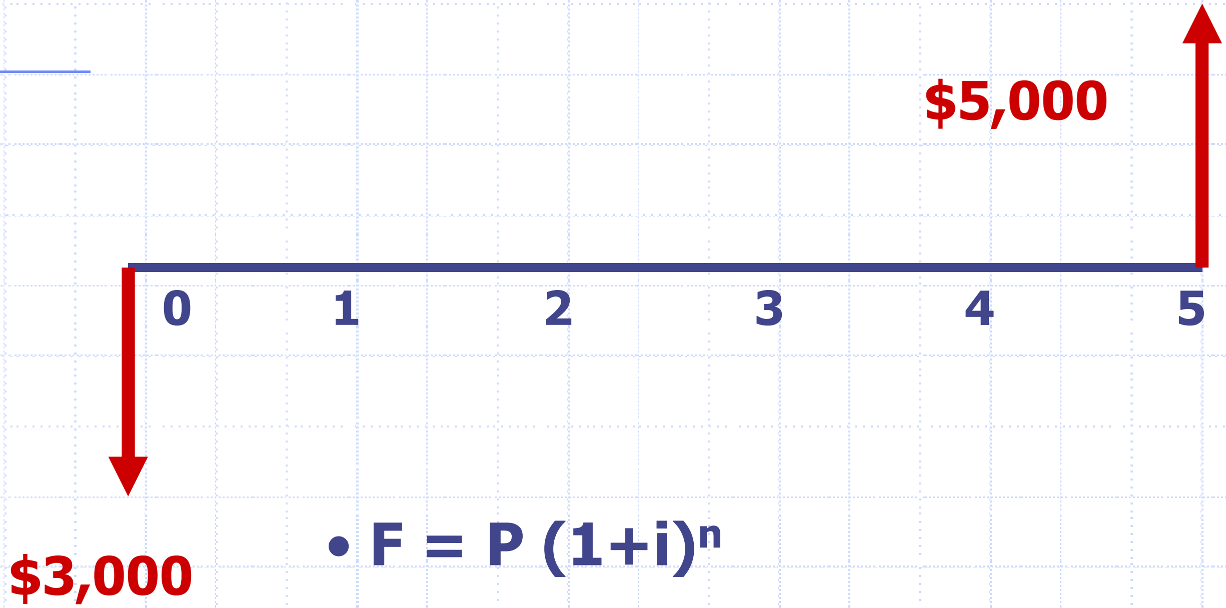
- In some problems, all of the parameters may be known except for the interest rate:
  - For example, “At what interest rate is this future stream of payments worth the same as $1000 today?”
- Solving for the unknown interest rate is called **“rate-of-return analysis”**
- Sometimes, we can solve for i in closed form
- In other cases (e.g., with annual payments):
  - **Interpolation** or **trial and error** must be used

Example 2.12 (from the book)

- Assume you can invest $3000 now in a friend's business, and will get paid back $5,000 in 5 years:
  - For what interest rate or "**internal rate of return" (IRR)** are these amounts equivalent?



Example 2.12 (continued)



- $F = P(1+i)^n$
- $5{,}000 = 3{,}000(1+i)^5$
- $(1+i)^5 = 5{,}000/3000 = 1.6667$
- $(1+i) = 1.6667^{0.20} = 1.1076$
- $i = 1.1076 - 1 = 0.1076 =$ **10.76%**
- This project is equivalent to earning 10.76% per year

Example of the IRR Function

Blank: Example 2.1 IRR Example Ex 2.12

t	CF(t)
0	-$3,000.00
1	$0.00
2	$0.00
3	$0.00
4	$0.00
5	$5,000.00
IRR =	10.757%

Cell D13 is the IRR function

Geometric Gradient Functior

=IRR($D6:$D11)

When the Interest Rate i Is Unknown

- In this example, solving for i was simple:
  - Because the problem involved only P and F
- Usually, you will need some other approach:
  - **Trial and error** (try different values of i until you converge)
  - Use look-up tables at back of textbook, and **interpolate** **(needed for exams!)**
  - Use **IRR function** in Excel

When the Interest Rate i Is Unknown

- When annual payments are involved:
  - Interpolation or trial and error must be used

$$A = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right]$$

$$A = F\left[\frac{i}{(1+i)^n - 1}\right]$$

Review of Interpolation (Section 2.4)

Tables of Conversion Factors

- All engineering economics textbooks provide tables of the various conversion factors:
  - Usually in an appendix at the end of the text
- Refer to the back of your text for those tables

Typical Format for Conversion Factors

	Int. Rate..........	0.1000	FOR i=	10.000%	PER CENT		
	SINGLE PAYMENT FACTORS		================	UNIFORM SERIES FACTORS	================		
	COMPOUND	PRESENT	SINKING	COMPOUND	CAPITAL	PRESENT	
N	AMT. FACTOR	WORTH	FUND	AMOUNT	RECOVERY	WORTH	n
	F/P	P/F	A/F	F/A	A/P	P/A	
1	1.1000	0.9091	1.0000	1.0000	1.10000	0.9091	1
2	1.2100	0.8264	0.4762	2.1000	0.57619	1.7355	2
3	1.3310	0.7513	0.3021	3.3100	0.40211	2.4869	3
4	1.4641	0.6830	0.2155	4.6410	0.31547	3.1699	4
5	1.6105	0.6209	0.1638	6.1051	0.26380	3.7908	5
6	1.7716	0.5645	0.1296	7.7156	0.22961	4.3553	6
7	1.9487	0.5132	0.1054	9.4872	0.20541	4.8684	7
8	2.1436	0.4665	0.0874	11.4359	0.18744	5.3349	8
9	2.3579	0.4241	0.0736	13.5795	0.17364	5.7590	9
10	2.5937	0.3855	0.0627	15.9374	0.16275	6.1446	10

Interpolation is a Process of Estimation!

- The unknown interest rate that you want will typically not be included in those tables:
  - But you can **interpolate** between two tabulated values to **estimate** it
- Linear interpolation is not exact, because:
  - The conversion factors are non-linear!
- Therefore, interpolation can cause errors:
  - Typically from 2-5%
- Errors will be small when interpolating between values that are close to each other

Interpolation Example

- You are considering a project that costs $1000 in year 0, and pays $144 annually for 10 years
- You want the interest rate such that **P = $1000** is equivalent to **A = $144/year** for **n = 10 years**
- Dividing tells us that **A/P = .144**
- Use the look-up tables in the back of the book to find interest rates that give values **near** .144

Interpolation Example (continued)

- For i = 7%, we observe:

N	COMPOUND AMT. FACTOR	PRESENT WORTH	SINKING FUND	COMPOUND AMOUNT	CAPITAL RECOVERY
	F/P	P/F	A/F	F/A	A/P
10	1.9672	0.5083	0.0724	13.8164	0.14238

(A/P, 7%, 10) = **0.14238**

Interpolation Example (continued)

- For i = 8%, we observe:

N	COMPOUND AMT. FACTOR	PRESENT WORTH	SINKING FUND	COMPOUND AMOUNT	CAPITAL RECOVERY
	F/P	P/F	A/F	F/A	A/P
10	2.1589	0.4632	0.0690	14.4866	0.14903

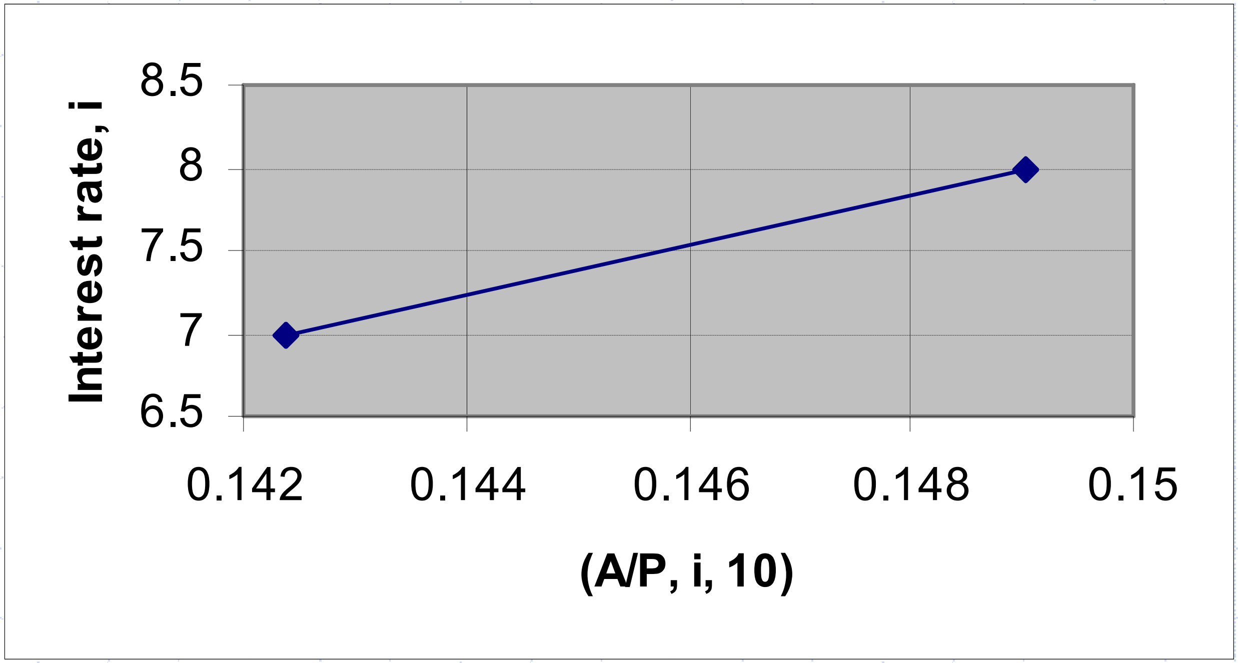
(A/P, 8%, 10) = <u>0.14903</u>

Interpolation Example (continued)

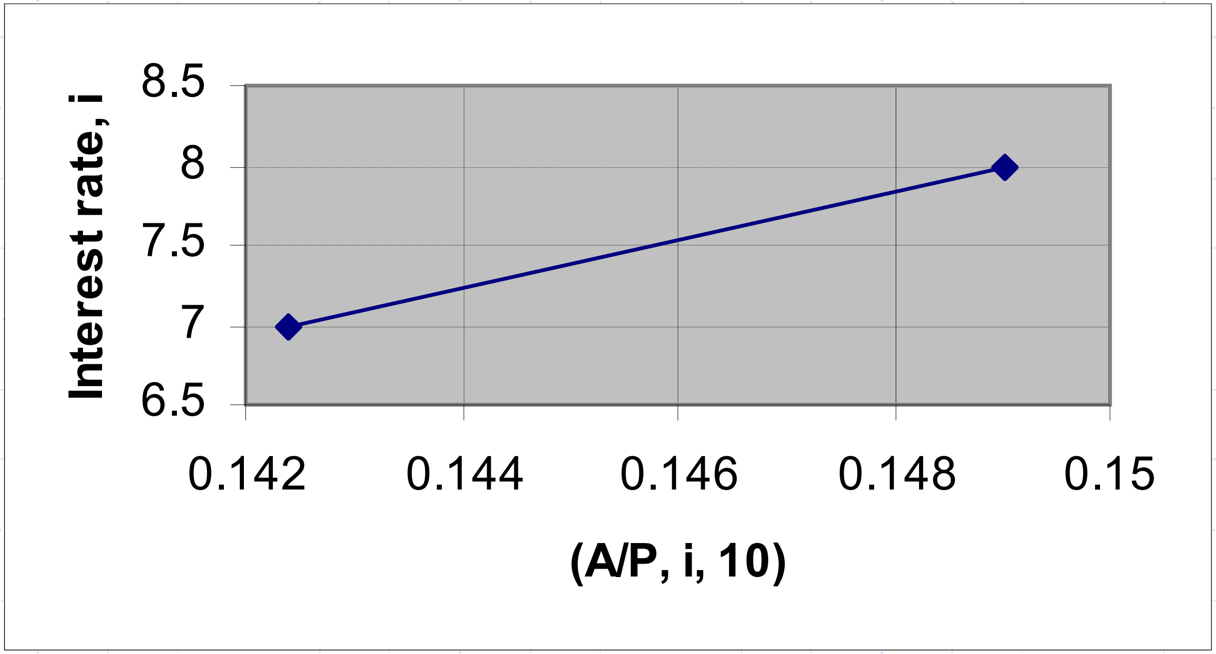
- Using the look-up tables, we found that:
  - At i = 7%, (A/P, 7%, 10) = .14238 (< .144)
  - At i = 8%, (A/P, 8%, 10) = .14903 (> .144)
- Therefore, we know that the IRR must be **between 7% and 8%**:
  - Could interpolate between 5% and 10%, but this would give bigger errors!

Interpolation Example (continued)

- Reading across at A/P = .144 lets us "eyeball" how close the IRR is to 7%
- Interpolation just lets us do that more exactly!

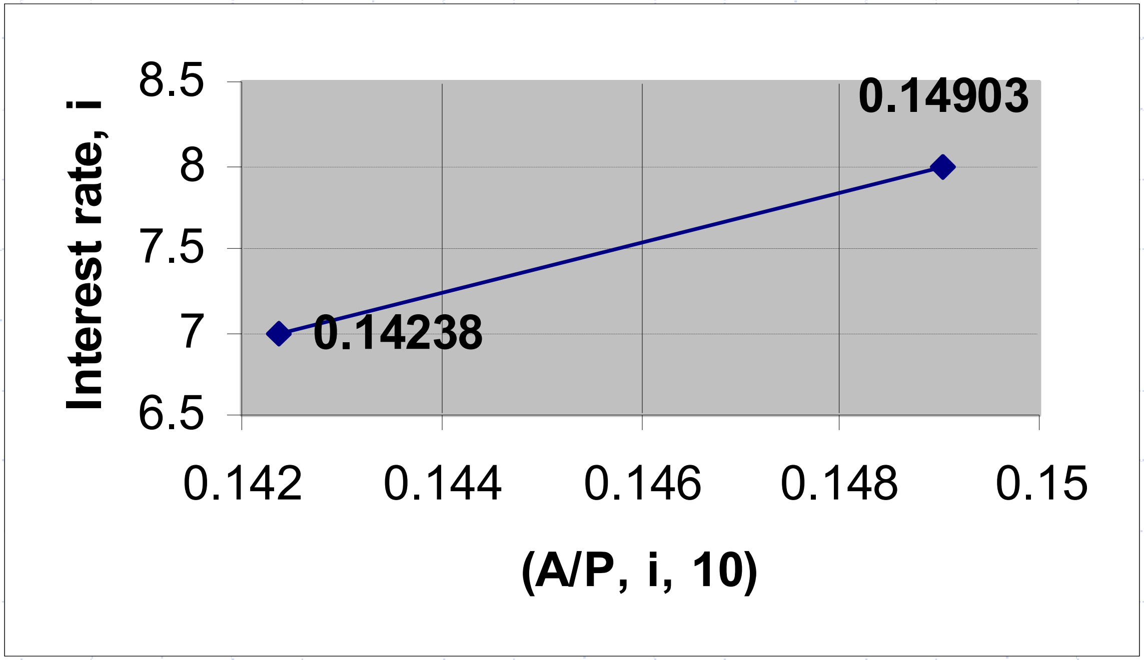


Interpolation Example (continued)



- **As the factor A/P increases, the interest rate i also increases**
- **Use the slope of this line to figure out how much to add to 7% to get A/P = .144**

Interpolation Example (continued)



- The formula for interpolation tells us:
- i ≈ 7%

+ [(8%-7%))/(.14903-.14238)] (.144-.14238)

- i ≈ 7.24%

Interpolation Example (continued)

- The formula for interpolation tells us:
- i ≈ 7%

+ [(8%-7%))/(.14903-.14238)] (.144-.14238)

- i ≈ 7.24%

- Using trial and error, the exact value of the IRR is 7.2459%

Interpolation Review

- Interpolation is ***approximate***, not exact!
- It requires:
  - One interest rate that gives a ***smaller*** ratio
  - One interest rate that gives a ***larger*** ratio
- The answer must lie between the two:
  - Results will be more accurate when the two starting points are close together

Compounding More Frequent than Annually

Compounding More than Annually

- Assume a time period:
  - Denoted by **t**
  - One year is standard
- Let **m** represent the number of times that interest is compounded within time **t**
- Then **t/m** is the "compounding period":
  - Normally, m = 1, and the compounding period is one year
  - But this need not be true!

Common Compounding Frequencies

- Every year – once a year (at the end):
  - (Annually)
- Every 6 months – 2 times a year:
  - (Semi-annually)
- Every quarter – 4 times a year:
  - (Quarterly)
- Every month – 12 times a year:
  - (Monthly)
- Every day – 365 times a year:
  - (Daily)
- Continuous – infinite number of compounding periods in a year!

Quotation of Interest Rates

- Interest rates can be quoted in several different ways
- Examples:
  - 12% per year
  - 1% per month
  - 12% per year, compounded monthly
- Thus, you have to “decipher” the various ways in order to determine:
  - Which interest rates are equivalent
  - Which interest rates are the best

Nominal and Effective Interest Rates

- **A nominal interest rate does not take into account the effects of compounding:**
  - Therefore, nominal rates can be misleading!
- **An effective interest rate is a true, periodic interest rate:**
  - That applies for a stated period of time
- **For example, with monthly compounding:**
  - 12% per year would be a *nominal interest rate*
  - 1% per month is the *effective interest rate*

Nominal Interest Rates

The term "nominal" means "in name only"

In other words, it is not the real interest rate!

Nominal and Effective Interest Rates

- We need a way to convert a ***nominal*** interest rate to the true ***effective interest rate*** that will actually apply!
- Mathematically, we can define the nominal interest rate **r** as:

r = (effective interest rate/period) (# of periods)

- So the ***effective interest rate*** can be computed as:

effective interest rate/period = r/(# of periods)

Examples of Nominal Interest Rates

- 1.5% per month effective interest rate:
  - Is the same as (1.5%) (12) = 18% nominal interest rate per year
  - Is the same as (1.5%) (6) = 9% nominal interest rate semiannually
- 1% per week effective interest rate:
  - Is the same as (1%) (52) = 52% nominal interest rate per year

Effective Interest Rates

- **An effective interest rate is a true, periodic interest rate:**
  - That applies for a stated period of time
- **It is conventional to use a year as the standard period of time:**
  - So, we would like to be able to convert a nominal interest rate to an effective annual interest rate

Effective Annual Interest Rate

- **Example:**
  - "*12% annual rate, compounded monthly*"
- **Pick this statement apart:**
  - **12% is the nominal interest rate**
  - **"Compounded monthly" tells us the number of compounding periods in a year (12)**
- **The effective interest rate per month is 1%:**
  - **We would like to be able to convert this to an effective annual interest rate**

Effective Annual Interest Rate

- The effective annual interest rate **i** for a nominal interest rate **r** compounded **m** times per year is:

$$i = (1 + r/m)^m - 1$$

Which One to Use: r or i

- **Some problems state only the nominal interest rate:**
  - The *nominal interest rate* is frequently stated for loans
  - Why?
- **Remember:**
  - Always use the effective interest rate in solving problems
  - (Either annual or per period)
- **The effective interest rate is always the one used in:**
  - Published interest tables
  - Closed-form time-value-of-money formulas
  - Spreadsheet functions

Monthly Compounding Example

- **Given:**

r = 9% per year, compounded monthly

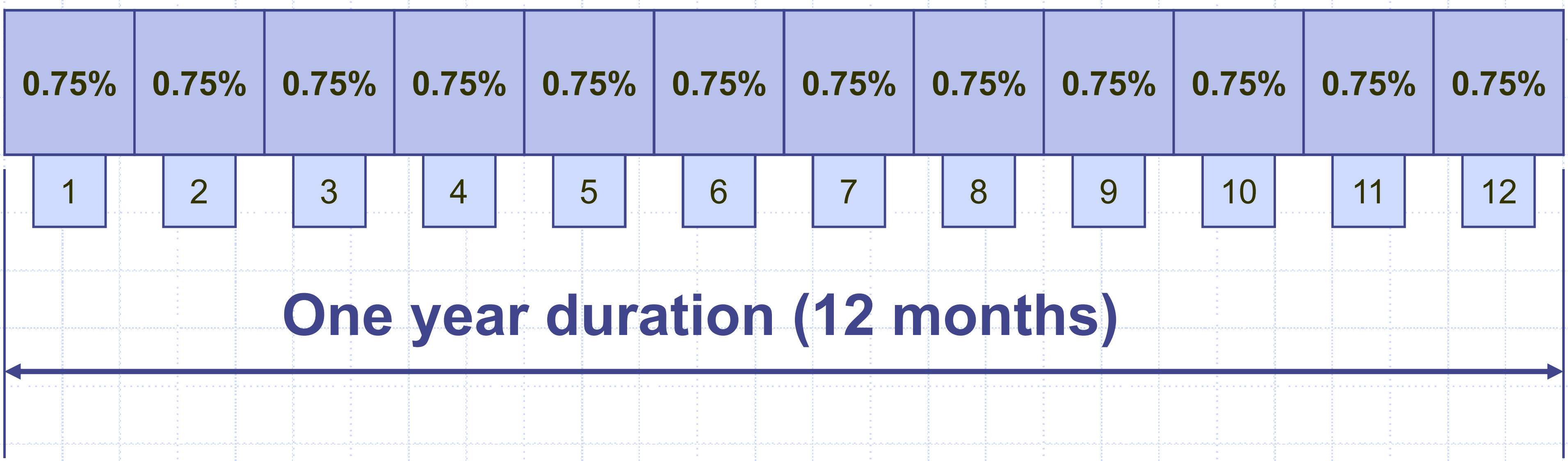
Compounding is monthly, so there are m = 12 compounding periods in a year

Effective monthly rate:
$0.09/12 = 0.0075 = 0.75\%/\text{month}$

Effective annual rate:
$(1 + 0.0075)^{12} - 1 = 0.0938 = 9.38\%/\text{year}$

Example (continued)

- r = 9% is the nominal rate
- "Compounded monthly" means **m = 12**
- The effective monthly rate is **0.75%/month**
- The **effective annual rate** is **9.38% per year**



Quarterly Compounding Example

- Given r = 9% per year, compounded quarterly

Quarter 1	Quarter 2	Quarter 3	Quarter 4

What is the **effective rate?**

- 0.09/4 = 0.0225 = 2.25%/quarter is the effective quarterly rate
- $(1 + .0225)^4 - 1 = 0.0930$ = 9.30%/year is the effective annual rate

Weekly Compounding Example

- **Given r = 9% per year, compounded weekly:**
  - Assume 52 weeks per year
  - The true effective weekly rate is (0.09/52) = 0.00173 = 0.173%/week
  - The effective annual rate is $(1 + 0.00173)^{52} - 1$ = 0.0940 = 9.40%/week

Comparison

- **The effective annual interest rate is always greater than the nominal interest rate:**
  - You are earning (paying) interest on your interest
- **The difference is greater with more frequent compounding:**
  - If compounded *quarterly*, we get 9.30%/year
  - If compounded *monthly*, we get 9.38%/year
  - If compounded *weekly*, we get 9.40%/year
- **What if we compound infinitely often?**

Continuous Compounding

- **Let the number of compounding periods get large:**

$$\lim_{m \to \infty} (1 + r/m)^m = e^r$$

Continuous Compounding

- To get **effective interest rate i** from **nominal interest rate r**:
  - $e^r = 1 + i \Rightarrow i = e^r - 1$
- To get **nominal interest rate r** from **effective interest rate i**:
  - $e^r = 1 + i \Rightarrow r = \ln(1 + i)$

Continuous Compounding

- **Why use continuous compounding?**
  - Some loans and investments are computed that way
  - Can be used to model revenue that comes in continuously during the year

Review

- **We learned how to solve for:**
  - Unknown number of years (analytic)
  - Unknown interest rate (numeric)
- **We learned about compounding:**
  - Discrete (important for this class)
  - Continuous