Lecture 4

Uncertain Interest Rates and Compounding
Overview

- Solving for unknown number of years
- Solving for unknown interest rate:
  - (Review of interpolation)
- Compounding more often than annually
Solving for an Unknown Number of Years
When the Number of Years $n$ Is Unknown

- Some problems require solving for a number of periods (NPER) given the other parameters
- We can always solve for $n$ in closed form...

\[ P = F \left[ \frac{1}{(1+i)^n} \right] \]

\[ A = P \left[ \frac{i(1+i)^n}{(1+i)^n - 1} \right] \]
Example 2.14 (from the book)

• How long will it take for $1,000 to double in value, at an interest rate of 5%?

\[ P = $1,000 \]
\[ F_n = $2000 \]
\[ i = 5\%/\text{year}; \ n \text{ is unknown!} \]
Example 2.14 (continued)

- \( F_n = 1000 \times (F/P, 5\%, n) \)
- \( 2000 = 1000 \times (1.05)^n \)
- We can solve for \( n \) in closed form...
Example 2.14 (continued)

- Solving, we get...
  - \((1.05)^n = \frac{2000}{1000}\)
  - \(n \ln(1.05) = \ln(2)\)
  - \(n = \frac{\ln(2)}{\ln(1.05)}\)
  - \(n = \frac{0.693147}{0.04879} = \textbf{14.2067 years}\)
- With compounding every year:
  - It will take \textbf{15 years} to amass $2,000
  - (Actually, a little more than $2,000!)
Example of the NPER Function

- In Excel, one can do this using NPER:

\[ \text{NPER}(i, A, P, F) \]

\[ =\text{NPER}(C23, C22, C20, C21) \]
Solving for an Unknown Interest Rate
When the Interest Rate $i$ Is Unknown

- In some problems, all of the parameters may be known except for the interest rate:
  - For example, “At what interest rate is this future stream of payments worth the same as $1000$ today?”
  - Solving for the unknown interest rate is called “rate-of-return analysis”
- Sometimes, we can solve for $i$ in closed form
- In other cases (e.g., with annual payments):
  - Interpolation or trial and error must be used
Example 2.12 (from the book)

- Assume you can invest $3000 now in a friend’s business, and will get paid back $5,000 in 5 years:

- For what interest rate or “internal rate of return” (IRR) are these amounts equivalent?
Example 2.12 (continued)

- \( F = P (1+i)^n \)
- \( 5000 = 3000 (1+i)^5 \)
- \( (1+i)^5 = \frac{5000}{3000} = 1.6667 \)
- \( (1+i) = 1.6667^{0.20} = 1.1076 \)
- \( i = 1.1076 - 1 = 0.1076 = 10.76\% \)
- This project is equivalent to earning 10.76% per year
Example of the IRR Function

<table>
<thead>
<tr>
<th>t</th>
<th>CF(t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-$3,000.00</td>
</tr>
<tr>
<td>1</td>
<td>$0.00</td>
</tr>
<tr>
<td>2</td>
<td>$0.00</td>
</tr>
<tr>
<td>3</td>
<td>$0.00</td>
</tr>
<tr>
<td>4</td>
<td>$0.00</td>
</tr>
<tr>
<td>5</td>
<td>$5,000.00</td>
</tr>
</tbody>
</table>

=IRR($D6:$D11)

Cell D13 is the IRR function
When the Interest Rate \( i \) Is Unknown

- In this example, solving for \( i \) was simple:
  - Because the problem involved only \( P \) and \( F \)
- Usually, you will need some other approach:
  - Trial and error (try different values of \( i \) until you converge)
  - Use look-up tables at back of textbook, and interpolate (needed for exams!)
  - Use IRR function in Excel
When the Interest Rate $i$ Is Unknown

- **When annual payments are involved:**
  - **Interpolation or trial and error must be used**

\[
A = P \left[ \frac{i(1+i)^n}{(1+i)^n - 1} \right]
\]

\[
A = F \left[ \frac{i}{(1+i)^n - 1} \right]
\]
Review of Interpolation (Section 2.4)
Tables of Conversion Factors

- All engineering economics textbooks provide tables of the various conversion factors:
  - Usually in an appendix at the end of the text
  - Refer to the back of your text for those tables
## Typical Format for Conversion Factors

<table>
<thead>
<tr>
<th>N</th>
<th>Int. Rate........ 0.1000</th>
<th>FOR i = 10.000% PER CENT</th>
<th>SINGLE PAYMENT FACTORS</th>
<th>~~~~~~~~~~~~ UNIFORM SERIES FACTORS ~~~~~~~~~~~~</th>
</tr>
</thead>
<tbody>
<tr>
<td>N</td>
<td></td>
<td></td>
<td>COMPOUND</td>
<td>PRESENT AMT. FACTOR</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>F/P</td>
<td>P/F</td>
</tr>
<tr>
<td>1</td>
<td>1.1000</td>
<td></td>
<td>0.9091</td>
<td>1.0000</td>
</tr>
<tr>
<td>2</td>
<td>1.2100</td>
<td></td>
<td>0.8264</td>
<td>0.4762</td>
</tr>
<tr>
<td>3</td>
<td>1.3310</td>
<td></td>
<td>0.7513</td>
<td>0.3021</td>
</tr>
<tr>
<td>4</td>
<td>1.4641</td>
<td></td>
<td>0.6830</td>
<td>0.2155</td>
</tr>
<tr>
<td>5</td>
<td>1.6105</td>
<td></td>
<td>0.6209</td>
<td>0.1638</td>
</tr>
<tr>
<td>6</td>
<td>1.7716</td>
<td></td>
<td>0.5645</td>
<td>0.1296</td>
</tr>
<tr>
<td>7</td>
<td>1.9487</td>
<td></td>
<td>0.5132</td>
<td>0.1054</td>
</tr>
<tr>
<td>8</td>
<td>2.1436</td>
<td></td>
<td>0.4665</td>
<td>0.0874</td>
</tr>
<tr>
<td>9</td>
<td>2.3579</td>
<td></td>
<td>0.4241</td>
<td>0.0736</td>
</tr>
<tr>
<td>10</td>
<td>2.5937</td>
<td></td>
<td>0.3855</td>
<td>0.0627</td>
</tr>
</tbody>
</table>
Interpolation is a Process of Estimation!

- The unknown interest rate that you want will typically not be included in those tables:
  - But you can **interpolate** between two tabulated values to **estimate** it
- Linear interpolation is not exact, because:
  - The conversion factors are non-linear!
- Therefore, interpolation can cause errors:
  - Typically from 2-5%
- Errors will be small when interpolating between values that are close to each other
Interpolation Example

- You are considering a project that costs $1000 in year 0, and pays $144 annually for 10 years.
- You want the interest rate such that $P = 1000$ is equivalent to $A = 144/\text{year}$ for $n = 10$ years.
- Dividing tells us that $A/P = 0.144$.
- Use the look-up tables in the back of the book to find interest rates that give values near 0.144.
For $i = 7\%$, we observe:

<table>
<thead>
<tr>
<th>N</th>
<th>COMPOUND AMT. FACTOR</th>
<th>PRESENT WORTH</th>
<th>SINKING FUND</th>
<th>COMPOUND AMOUNT</th>
<th>CAPITAL RECOVERY</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>F/P</td>
<td>P/F</td>
<td>A/F</td>
<td>F/A</td>
<td>A/P</td>
</tr>
<tr>
<td></td>
<td>1.9672</td>
<td>0.5083</td>
<td>0.0724</td>
<td>13.8164</td>
<td>0.14238</td>
</tr>
</tbody>
</table>

$(A/P, 7\%, 10) = 0.14238$
Interpolation Example (continued)

For $i = 8\%$, we observe:

<table>
<thead>
<tr>
<th>N</th>
<th>COMPOUND AMT. FACTOR</th>
<th>PRESENT WORTH</th>
<th>SINKING FUND</th>
<th>COMPOUND AMOUNT</th>
<th>CAPITAL RECOVERY</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>F/P 2.1589</td>
<td>P/F 0.4632</td>
<td>A/F 0.0690</td>
<td>F/A 14.4866</td>
<td>A/P 0.14903</td>
</tr>
</tbody>
</table>

$(A/P, 8\%, 10) = 0.14903$
Using the look-up tables, we found that:

- At $i = 7\%$, $(A/P, 7\%, 10) = 0.14238 (< 0.144)$
- At $i = 8\%$, $(A/P, 8\%, 10) = 0.14903 (> 0.144)$

Therefore, we know that the IRR must be between 7\% and 8\%:

- Could interpolate between 5\% and 10\%, but this would give bigger errors!
Interpolation Example (continued)

- Reading across at $A/P = 0.144$ lets us “eyeball” how close the IRR is to 7%.
- Interpolation just lets us do that more exactly!
As the factor $A/P$ increases, the interest rate $i$ also increases.

Use the slope of this line to figure out how much to add to 7% to get $A/P = 0.144$. 
Interpolation Example (continued)

- The formula for interpolation tells us:
- \( i \approx 7\% \) 
  \[ + \frac{[(8\%-7\%)]}{(0.14903-0.14238)} \times (0.144-0.14238) \]
- \( i \approx 7.24\% \)
Interpolation Example (continued)

• The formula for interpolation tells us:
  \[ i \approx 7\% + \frac{(8\%-7\%)}{(.14903-.14238)} (.144-.14238) \]
  \[ i \approx 7.24\% \]

• Using trial and error, the exact value of the IRR is 7.2459\%
Interpolation Review

Interpolation is *approximate*, not exact!

It requires:

- One interest rate that gives a *smaller* ratio
- One interest rate that gives a *larger* ratio

The answer must lie between the two:

- Results will be more accurate when the two starting points are close together
Compounding More Frequent than Annually
Compounding More than Annually

- Assume a time period:
  - Denoted by $t$
  - One year is standard
- Let $m$ represent the number of times that interest is compounded within time $t$
- Then $t/m$ is the “compounding period”:
  - Normally, $m = 1$, and the compounding period is one year
  - But this need not be true!
Common Compounding Frequencies

- **Every year** – once a year (at the end):
  - (Annually)
- **Every 6 months** – 2 times a year:
  - (Semi-annually)
- **Every quarter** – 4 times a year:
  - (Quarterly)
- **Every month** – 12 times a year:
  - (Monthly)
- **Every day** – 365 times a year:
  - (Daily)
- **Continuous** – infinite number of compounding periods in a year!
Quotation of Interest Rates

- Interest rates can be quoted in several different ways

Examples:
- 12% per year
- 1% per month
- 12% per year, compounded monthly

Thus, you have to “decipher” the various ways in order to determine:
- Which interest rates are equivalent
- Which interest rates are the best
Nominal and Effective Interest Rates

- A nominal interest rate does not take into account the effects of compounding:
  - Therefore, nominal rates can be misleading!
- An effective interest rate is a true, periodic interest rate:
  - That applies for a stated period of time
- For example, with monthly compounding:
  - 12% per year would be a nominal interest rate
  - 1% per month is the effective interest rate
Nominal Interest Rates

The term “nominal” means “in name only”

In other words, it is not the real interest rate!
Nominal and Effective Interest Rates

- We need a way to convert a nominal interest rate to the true effective interest rate that will actually apply!
- Mathematically, we can define the nominal interest rate $r$ as:

\[ r = \text{(effective interest rate/period)} \times \text{(number of periods)} \]

- So the effective interest rate can be computed as:

\[ \text{effective interest rate/period} = \frac{r}{\text{(number of periods)}} \]
Examples of Nominal Interest Rates

- **1.5% per month effective interest rate:**
  - Is the same as \((1.5\%) \times (12) = 18\%\) nominal interest rate per year
  - Is the same as \((1.5\%) \times (6) = 9\%\) nominal interest rate semiannually

- **1% per week effective interest rate:**
  - Is the same as \((1\%) \times (52) = 52\%\) nominal interest rate per year
Effective Interest Rates

- An **effective interest rate** is a true, periodic interest rate:
  - That applies for a stated period of time
- **It is conventional to use a year as the standard period of time:**
  - So, we would like to be able to convert a nominal interest rate to an **effective annual interest rate**
Example:

“12% annual rate, compounded monthly”

Pick this statement apart:

- 12% is the nominal interest rate
- “Compounded monthly” tells us the number of compounding periods in a year (12)

The effective interest rate per month is 1%:

- We would like to be able to convert this to an effective annual interest rate
Effective Annual Interest Rate

The effective annual interest rate $i$ for a nominal interest rate $r$ compounded $m$ times per year is:

\[ i = (1 + \frac{r}{m})^m - 1 \]
Some problems state only the nominal interest rate:
- The **nominal interest rate** is frequently stated for loans
- Why?

**Remember:**
- Always use the **effective interest rate** in solving problems
- (Either annual or per period)

**The effective interest rate is always the one used in:**
- Published interest tables
- Closed-form time-value-of-money formulas
- Spreadsheet functions
Monthly Compounding Example

Given:

\[ r = 9\% \text{ per year, compounded monthly} \]

Compounding is monthly, so there are \( m = 12 \) compounding periods in a year.

Effective monthly rate:
\[
0.09/12 = 0.0075 = 0.75\%/\text{month}
\]

Effective annual rate:
\[
(1 + 0.0075)^{12} - 1 = 0.0938 = 9.38\%/\text{year}
\]
Example (continued)

- \( r = 9\% \) is the nominal rate
- “Compounded monthly” means \( m = 12 \)
- The effective monthly rate is 0.75%/month
- The effective annual rate is 9.38% per year
Quarterly Compounding Example

Given \( r = 9\% \) per year, compounded quarterly

<table>
<thead>
<tr>
<th>Quarter 1</th>
<th>Quarter 2</th>
<th>Quarter 3</th>
<th>Quarter 4</th>
</tr>
</thead>
</table>

What is the effective rate?

\[ 0.09/4 = 0.0225 = 2.25\%/\text{quarter} \text{ is the effective quarterly rate} \]

\[ (1 + 0.0225)^4 - 1 = 0.0930 = 9.30\%/\text{year} \text{ is the effective annual rate} \]
Weekly Compounding Example

Given \( r = 9\% \) per year, compounded weekly:

- Assume 52 weeks per year
- The true effective weekly rate is \( \frac{0.09}{52} = 0.00173 = 0.173\%/\text{week} \)
- The effective annual rate is \( (1 + 0.00173)^{52} - 1 = 0.0940 = 9.40\%/\text{week} \)
The effective annual interest rate is always greater than the nominal interest rate:
- You are earning (paying) interest on your interest

The difference is greater with more frequent compounding:
- If compounded \textit{quarterly}, we get \textbf{9.30\%/year}
- If compounded \textit{monthly}, we get \textbf{9.38\%/year}
- If compounded \textit{weekly}, we get \textbf{9.40\%/year}

What if we compound \textit{infinitely} often?
Continuous Compounding

Let the number of compounding periods get large:

\[
\lim_{m \to \infty} (1 + \frac{r}{m})^m = e^r
\]
Continuous Compounding

To get effective interest rate $i$ from nominal interest rate $r$:
\[ e^r = 1 + i \implies i = e^r - 1 \]

To get nominal interest rate $r$ from effective interest rate $i$:
\[ e^r = 1 + i \implies r = \ln(1 + i) \]
Continuous Compounding

Why use continuous compounding?

- Some loans and investments are computed that way.
- Can be used to model revenue that comes in continuously during the year.
We learned how to solve for:
- Unknown number of years (analytic)
- Unknown interest rate (numeric)

We learned about compounding:
- Discrete (important for this class)
- Continuous