Lecture 4

Uncertain Interest Rates and Compounding
Uncertain interest rates

- It may be of interest to determine the value of $i$ that makes two streams of payments equivalent

Why?

- Compare a series of payments to a fixed interest rate
- Determine attractiveness of the payments
- Provide a simple summary measure
Let’s say we’re given $F$ and $P$:
- E.g., lend $P$ now, receive $F$ in year $n$

We know that $F = P(1+i)^n$

Can solve for $i$ easily:
- $F/P = (1+i)^n \implies (F/P)^{1/n} = 1 + i \implies i = (F/P)^{1/n} - 1$

Can solve for $n$ also, using logarithms:
- $\log(F) = \log(P) + n \log(i)$
- $n = [\log(F) - \log(P)]/\log(i)$
Uncertain interest rates

- If there is an annual series of payments, this gets more complicated:

\[ P = \sum_{t=1}^{n} \frac{A}{(1 + i)^t} = A\left[1 - \frac{1}{(1 + i)^n}\right]/i \]

- Can find \( n \) (using logarithms), but not \( i \)!
  - Because \( i \) appears \textit{twice}:
    - Once with an exponent
    - Once without
Uncertain interest rates

- How to solve?

- Two options
  - Trial and error:
    - Try different values of $i$ until you converge
    - I.e., until $P = A \left[ 1 - \frac{1}{(1+i)^n} \right]/i$
  - Use look-up tables
    - (e.g., tables at back of textbook) and interpolate
Example of interpolation

- We are told that $P = $100 is equivalent to payment of $A = $20/year for 6 years
- Dividing tells us that $A/P = .2$
  - And we know that $n = 6$
- For $i = 5\%$, we find $A/P = .19702 < .2$
  - For $i = 6\%$, $A/P = .20336 > .2$
- The answer must lie between 5\% and 6\%
Example of interpolation

- For $i = 5\%$, we find $A/P = .19702 < .2$
- For $i = 6\%$, $A/P = .20336 > .2$
- The formula for interpolation tells us:
  - $i \approx 5\%
  + (6\%-5\%) (.2-.19702)/(.20336-.19702)$
  - $i \approx 5.47\%$
Interpolation

- Interpolation is *approximate*, not exact!
- It requires:
  - One interest rate that gives a *smaller* ratio
  - One interest rate that gives a *larger* ratio
- The answer must lie between the two:
  - Results will be more accurate when the two starting points are close together
Compounding--example

- r=12% interest, compounded monthly, is equivalent to 1% *monthly* interest
  - Borrow $1000 for 1 year, at 12% interest:
    - \[ F = P \left(1 + r\right) = 1.12 \times P = 1120 \]
  - If compounded *monthly*, we get:
    - \[ F = P \left(1 + \frac{r}{12}\right)^{12} = 1126.8 \]
  - The compounded amount is *greater!*
Compounding--example

- Difference is greater over longer period, or more frequent compounding
  - Borrow $1000 for 5 years, at 12% interest:
    - $F = P (1 + r)^5 = 1762.3$
  - If compounded *monthly*, we get:
    - $F = P (1 + r/12)^{60} = 1816.7$
  - The compounded amount is now *more than 3%* greater
General formula

- Interest compounded m times per year, at a rate \( r/m \) per period:
  - **Nominal** interest rate = \( r \)
  - **Effective** interest rate = \( i = (1 + r/m)^m - 1 \)
    - In other words, \( (1 + r/m)^m = 1 + i \)

- The *nominal* interest rate is frequently the value stated, especially for loans:
  - Why?
Continuous compounding

- Let the number of compounding periods get large:
  
  \[
  \lim_{m \to \infty} (1 + \frac{r}{m})^m = e^r
  \]

- Future value of P at nominal interest r, compounded continuously for n years:
  
  \[ F = P e^{rn} \]
Continuous compounding

- To get **effective** annual interest rate $i$ from nominal rate $r$:
  - $e^{rn} = (1 + i)^n$
  - $e^r = 1 + i$
  - $i = e^r - 1$

- To get **nominal** annual interest rate $r$ from effective rate $i$:
  - $e^r = 1 + i \Rightarrow r = \ln(1 + i)$
Continuous compounding

Why use continuous compounding?

- Some loans and investments are computed that way
- Can also be used to model revenue that comes in continuously throughout the year
Review

- We learned how to find uncertain interest rates
- We learned about compounding:
  - Discrete (more important for this class)
  - Continuous