Lecture 4

Uncertain Interest Rates
and Compounding

Uncertain interest rates

- It may be of interest to determine the value of i that makes two streams of payments equivalent
- Why?
  - Compare a series of payments to a fixed interest rate
  - Determine attractiveness of the payments
  - Provide a simple summary measure

Uncertain interest rates

- Let's say we're given F and P:
  - E.g., lend P now, receive F in year n
- We know that $F = P(1+i)^n$
- Can solve for i easily:
  - $F/P = (1+i)^n \Rightarrow (F/P)^{1/n} = 1 + i \Rightarrow i = (F/P)^{1/n} - 1$
- Can solve for n also, using logarithms:
  - $\log(F) = \log(P) + n \log(i)$
  - $n = [\log(F) - \log(P)]/\log(i)$

Uncertain interest rates

- If there is an annual series of payments, this gets more complicated:

$$P = \sum_{t=1}^{n} A/(1+i)^{t} = A[1-1/(1+i)^{n}]/i$$

- Can find n (using logarithms), but *not* i!
  - Because i appears *twice*:
    - Once with an exponent
    - Once without

Uncertain interest rates

- How to solve?
- Two options
  - Trial and error:
    - Try different values of i until you converge
    - I.e., until $P = A [1 - 1/(1+i)^n]/i$
  - Use look-up tables
    - (e.g., tables at back of textbook)

  and interpolate

Example of interpolation

- We are told that P = $100 is equivalent to payment of A = $20/year for 6 years
- Dividing tells us that A/P = .2
  - And we know that n = 6
- For i = 5%, we find A/P = .19702 < .2
  - For i = 6%, A/P = .20336 > .2
  - The answer must lie between 5% and 6%

Example of interpolation

- For $i = 5\%$, we find $A/P = .19702 < .2$
- For $i = 6\%$, $A/P = .20336 > .2$
- The formula for interpolation tells us:
  - $i \approx 5\%$
    $+ (6\%-5\%)\,(.2-.19702)/(.20336-.19702)$
  - $i \approx 5.47\%$

Interpolation

- Interpolation is *approximate*, not exact!
- It requires:
  - One interest rate that gives a *smaller* ratio
  - One interest rate that gives a *larger* ratio
- The answer must lie between the two:
  - Results will be more accurate when the two starting points are close together

Compounding--example

- r=12% interest, compounded monthly, is equivalent to 1% *monthly* interest
  - Borrow $1000 for 1 year, at 12% interest:
    - $F = P (1 + r) = 1.12 P = 1120$
  - If compounded *monthly*, we get:
    - $F = P (1 + r/12)^{12} = 1126.8$
  - The compounded amount is *greater!*

Compounding--example

- Difference is greater over longer period, or more frequent compounding
  - Borrow \$1000 for 5 years, at 12% interest:
    - $F = P\,(1 + r)^5 = 1762.3$
  - If compounded *monthly*, we get:
    - $F = P\,(1 + r/12)^{60} = 1816.7$
  - The compounded amount is now *more than 3%* greater

General formula

- Interest compounded m times per year, at a rate r/m per period:
  - *Nominal* interest rate = r
  - *Effective* interest rate = i = $(1 + r/m)^m - 1$
    - In other words, $(1 + r/m)^m = 1 + i$
- The *nominal* interest rate is frequently the value stated, especially for loans:
  - Why?

Continuous compounding

- Let the number of compounding periods get large:

$$\lim_{m\to\infty} (1 + r/m)^m = e^r$$

- Future value of P at nominal interest r, compounded continuously for n years:
  - $F = P\, e^{rn}$

Continuous compounding

- To get *effective* annual interest rate i from nominal rate r:
  - $e^{rn} = (1 + i)^n$
  - $e^r = 1 + i$
  - $i = e^r - 1$
- To get *nominal* annual interest rate r from effective rate i:
  - $e^r = 1 + i \Rightarrow r = \ln(1 + i)$

Continuous compounding

- Why use continuous compounding?
  - Some loans and investments are computed that way
  - Can also be used to model revenue that comes in continuously throughout the year

Review

- We learned how to find uncertain interest rates
- We learned about compounding:
  - Discrete (more important for this class)
  - Continuous