Lecture 3

Financial Mathematics

Note!

- **We will *<u>assume no inflation!</u>***
  - In the discussion that follows
  - (And for the next several weeks)

Notation

- **i = interest rate (per time period)**
- **n = # of time periods**
- **P = money at *present***
- **F = money in *future***
  - After n time periods
  - Equivalent to P now, at interest rate i
- **A = payment at end of each time period**
  - E.g., *annual*

Assumptions

- **Assume all cash flow occurs at the *end* of each time period**
  - For example, all year 1 payments are due on December 31 of year 1
- **The present is the end of period 0**

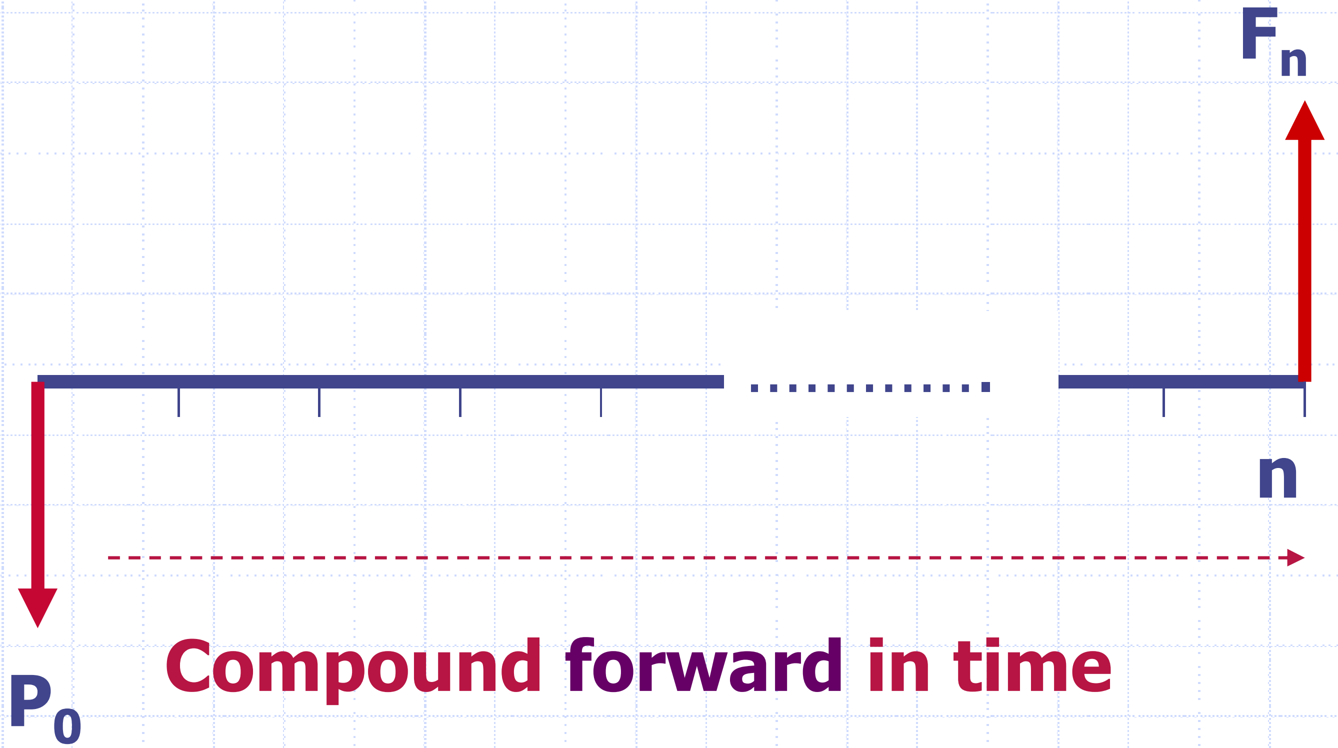
Overview

- Converting from P to F, and from F to P
- Converting from A to P, and from P to A
- Converting from F to A, and from A to F
- (No gradient methods!)
- **Sensitivity analysis** (Section 2.9)

Present to Future, and Future to Present

Converting from Present to Future

- **To find F given P:**



Derive by Recursion

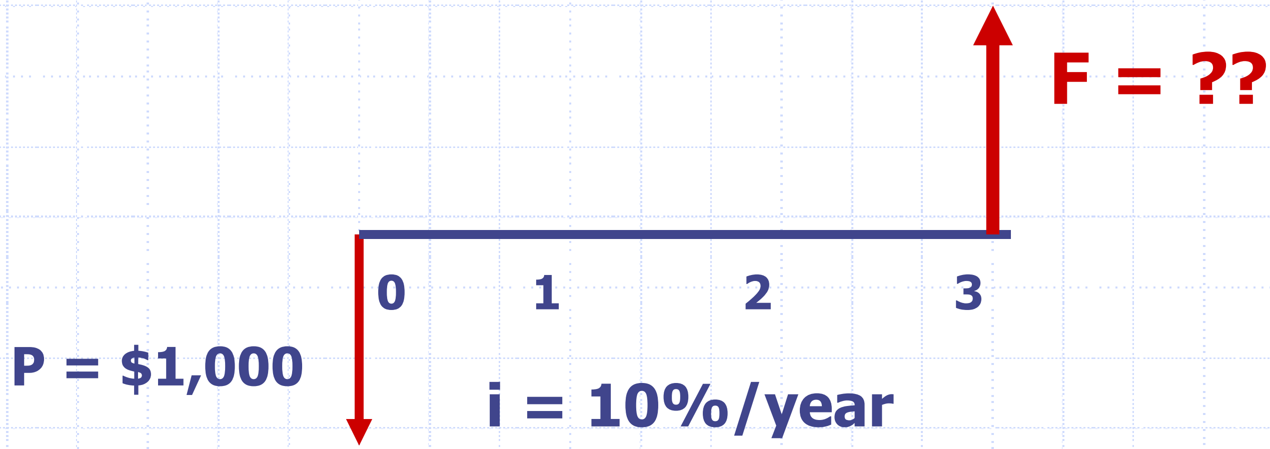
- **Invest an amount P at rate i:**
  - Amount at time 1 = P (1+i)
  - Amount at time 2 = $P(1+i)^2$
  - Amount at time n = $P(1+i)^n$
- **So we know that $F = P(1+i)^n$**
  - $(F/P, i\%, n) = (1+i)^n$
  - Single payment compound amount factor

$$F_n = P(1+i)^n$$

$$F_n = P\,(F/P, i\%, n)$$

Example—Present to Future

- Invest P=$1,000, n=3, i=10%
- What is the future value, F?

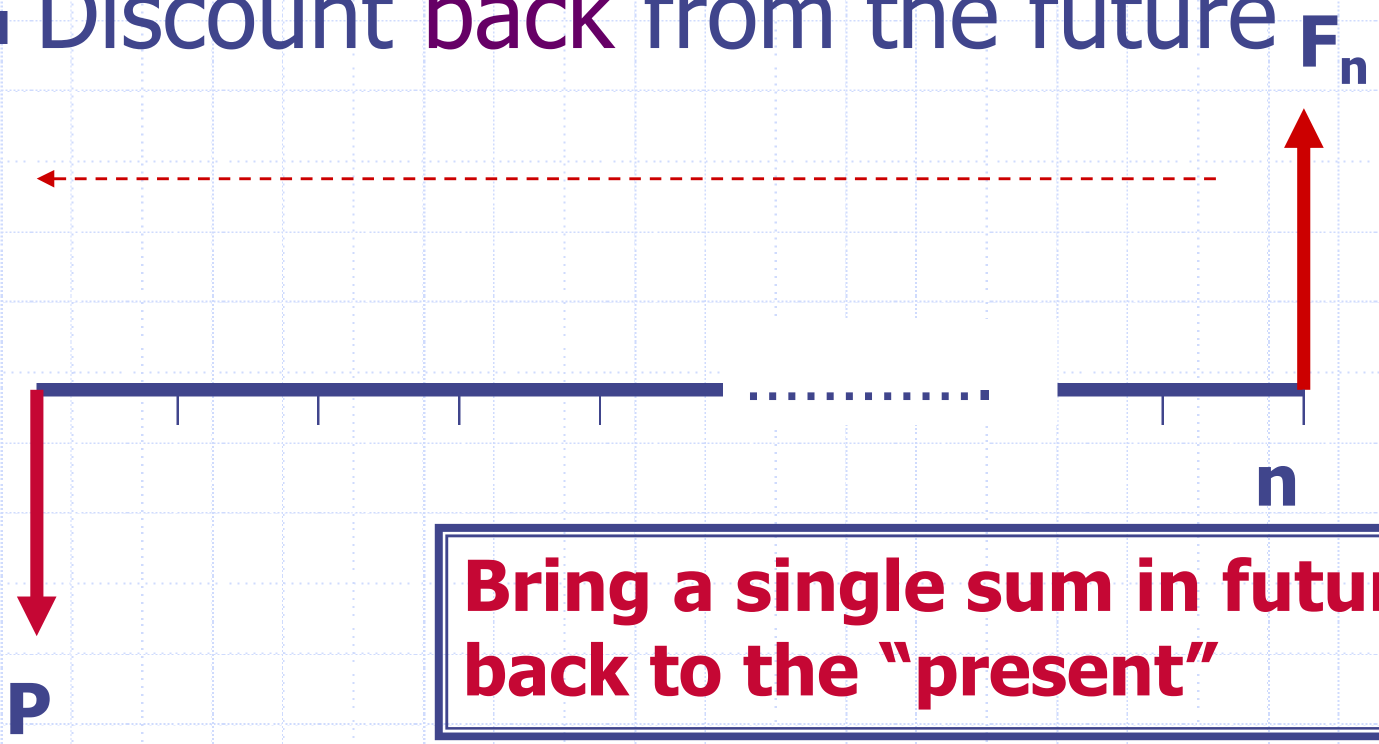


$$F_3 = \$1,000\ (F/P, 10\%, 3) = \$1,000\ (1.10)^3$$

$$= \$1,000\ (1.3310) = \underline{\$1,331.00}$$

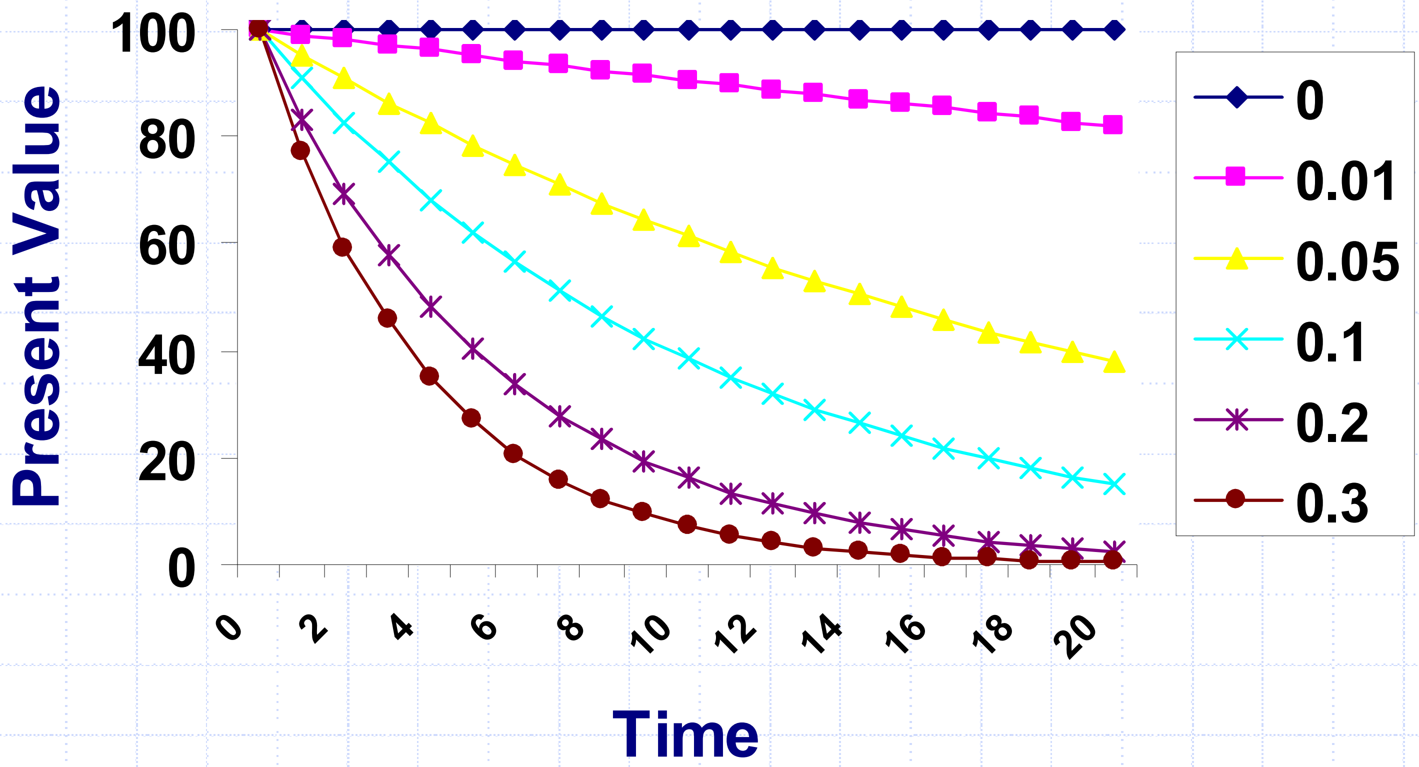
Converting from Future to Present

- **To find P given F:**
  - Discount back from the future



Bring a single sum in future back to the "present"

Illustration of Discounting

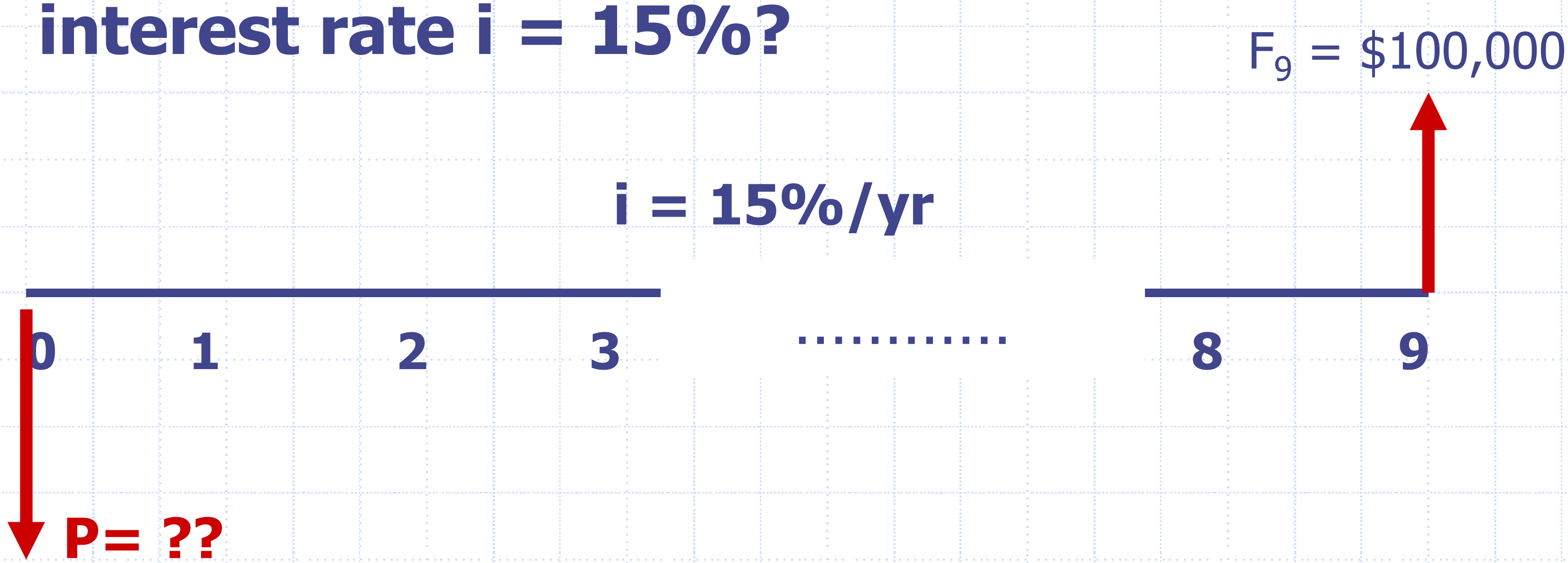


Converting from Future to Present

- **Amount F at time n:**
  - Amount at time n-1 = $F/(1+i)$
  - Amount at time n-2 = $F/(1+i)^2$
  - Amount at time 0 = $F/(1+i)^n$
- **So we know that $P = F/(1+i)^n$**
  - $(P/F, i\%, n) = 1/(1+i)^n$
  - Single payment present worth factor

Example—Future to Present

- **Assume we want F = $100,000 in 9 years.**
- **How much do we need to invest now, if the interest rate i = 15%?**



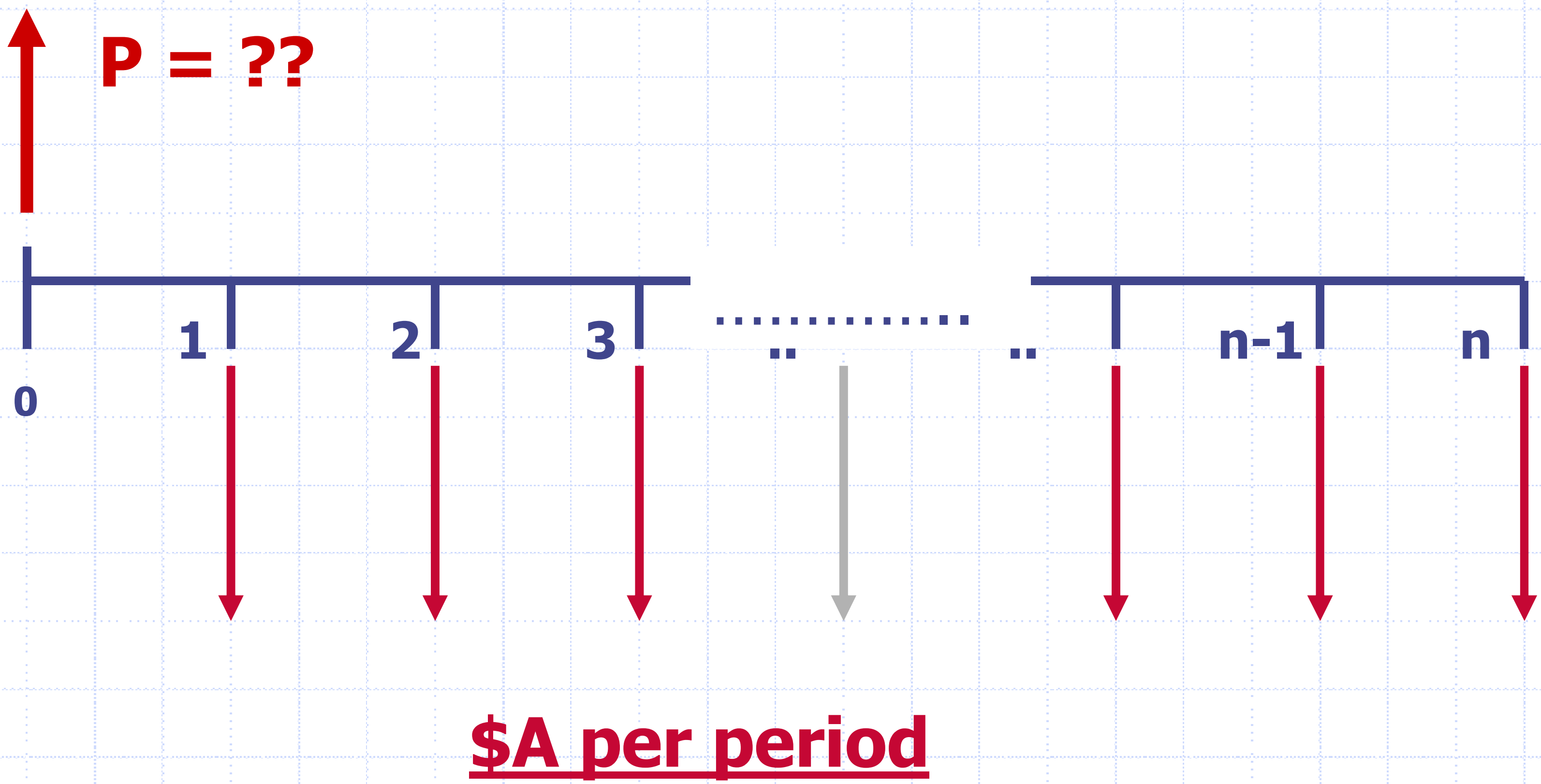
P = $100,000 (P/F, 15%, 9) = $100,000 $[1/(1.15)^9]$

= $100,000 (0.1111) = <u>$11,110</u> at time t = 0

Annual to Present, and Present to Annual

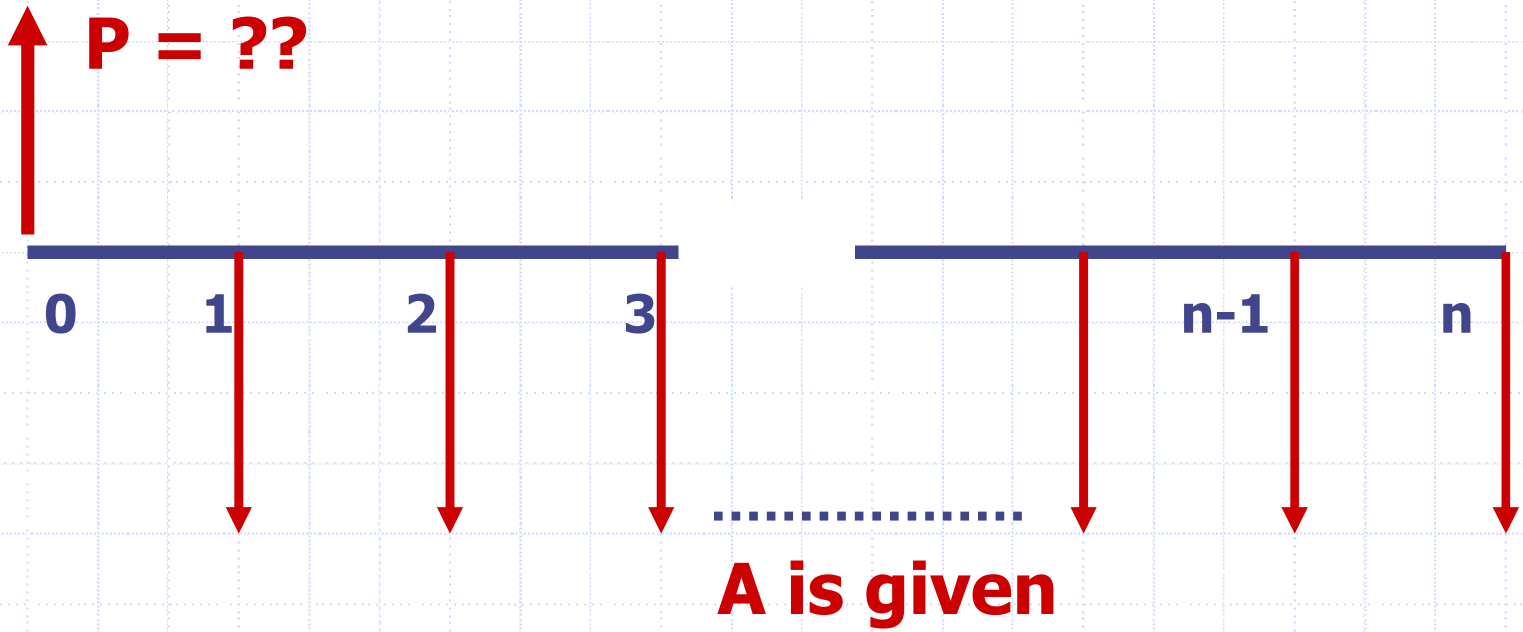
Converting from Annual to Present

- Fixed annuity—constant cash flow



Converting from Annual to Present

- We want an expression for the present worth **P** of a stream of equal, end-of-period cash flows **A**



Converting from Annual to Present

- **Write a present-worth expression for each year individually, and add them**

$$P = A\left[\frac{1}{(1+i)^1} + \frac{1}{(1+i)^2} + .. + \frac{1}{(1+i)^{n-1}} + \frac{1}{(1+i)^n}\right]$$

The term inside the brackets is a geometric progression.

This sum has a closed-form expression!

Converting from Annual to Present

- **Write a present-worth expression for each year individually, and add them**

$$P = A\left[\frac{1}{(1+i)^1} + \frac{1}{(1+i)^2} + .. + \frac{1}{(1+i)^{n-1}} + \frac{1}{(1+i)^n}\right]$$

$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] \quad for\ i \neq 0$$

(Derivation given in Section 2.2)

Converting from Annual to Present

- **This expression will convert an annual cash flow to an equivalent present worth amount:**
  - (One period before the first annual cash flow)

$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] \; for \; i \neq 0$$

- **The term in the brackets is (P/A, i%, n)**
- **Uniform series present worth factor**

Converting from Present to Annual

- **Given the P/A relationship:**

$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] \quad for\ i \neq 0$$

We can just solve for A in terms of P, yielding:

$$A = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right]$$

Remember: The present is always one period before the first annual amount!

- **The term in the brackets is (A/P, i%, n)**
- **Capital recovery factor**

Converting from Present to Annual

- **This is how mortgages and car loans work:**
  - The bank gives you an amount P today
  - You pay equal amounts A until you have paid the loan plus interest
  - In the first year, you pay mainly interest, and little of the principal
  - In the last year, you pay mainly the principal, and little interest (since little of your original loan amount P is still owed)

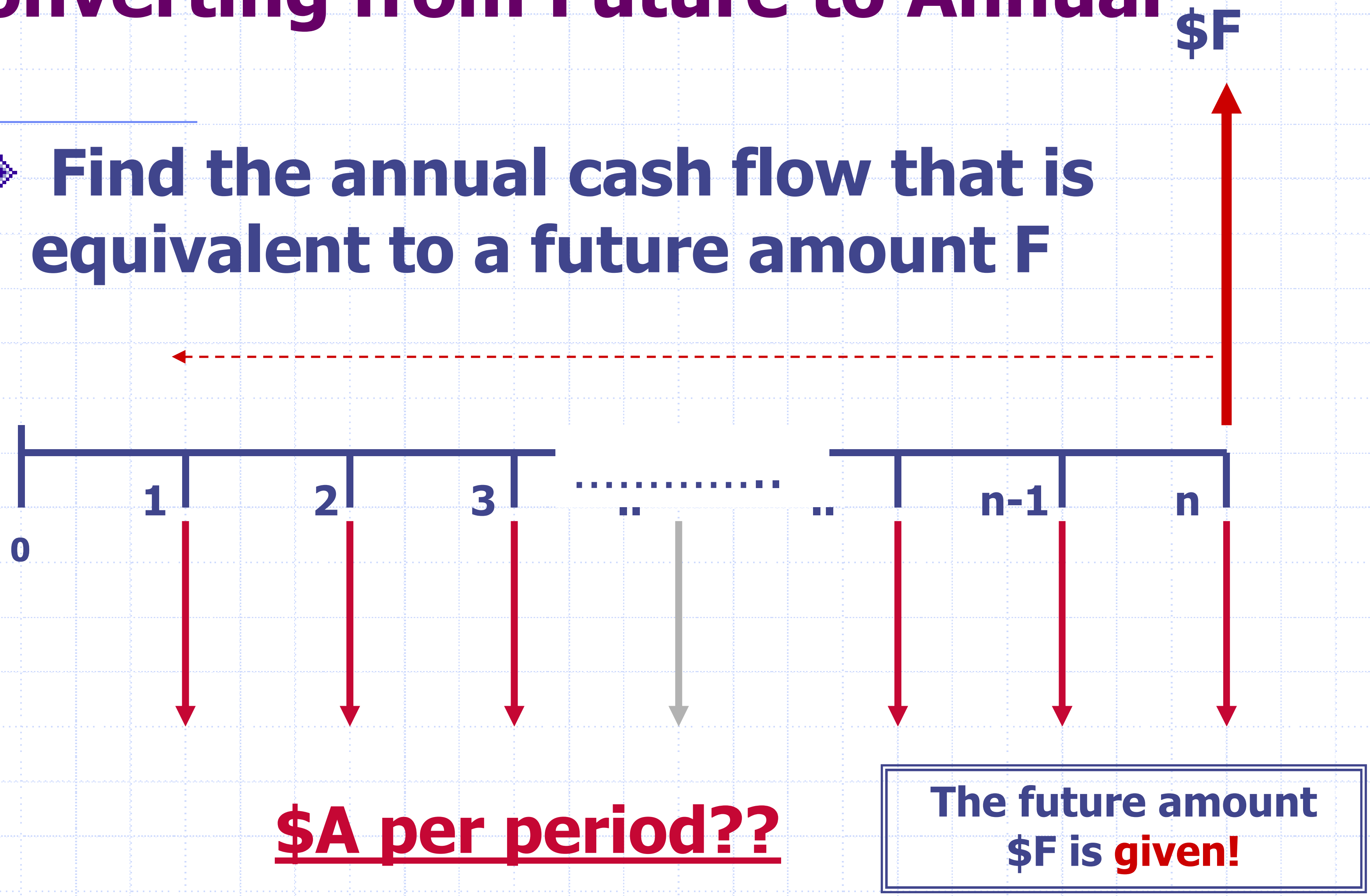
Converting from Present to Annual

- **How is it possible to calculate a constant amount to repay, and have the total be exactly equivalent to P?**
  - It is sort of like magic!
- **The calculations would be easier if you paid an equal fraction of the principal P every year, plus whatever interest is owed on the unpaid portion of the principal:**
  - But in that case almost nobody could afford to get a mortgage, because the payments would be very high in the first few years!

Future to Annual, and Annual to Future

Converting from Future to Annual

- Find the annual cash flow that is equivalent to a future amount F



Converting from Future to Annual

- Take advantage of what we know
- Recall that:

$$P = F\left[\frac{1}{(1+i)^n}\right]$$

Substitute "P" and simplify!

and

$$A = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right]$$

Converting from Future to Annual

- **First convert future to present:**
  - Then convert the resulting P to annual

$$A = F\left[\frac{1}{(1+i)^n}\right]\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right]$$

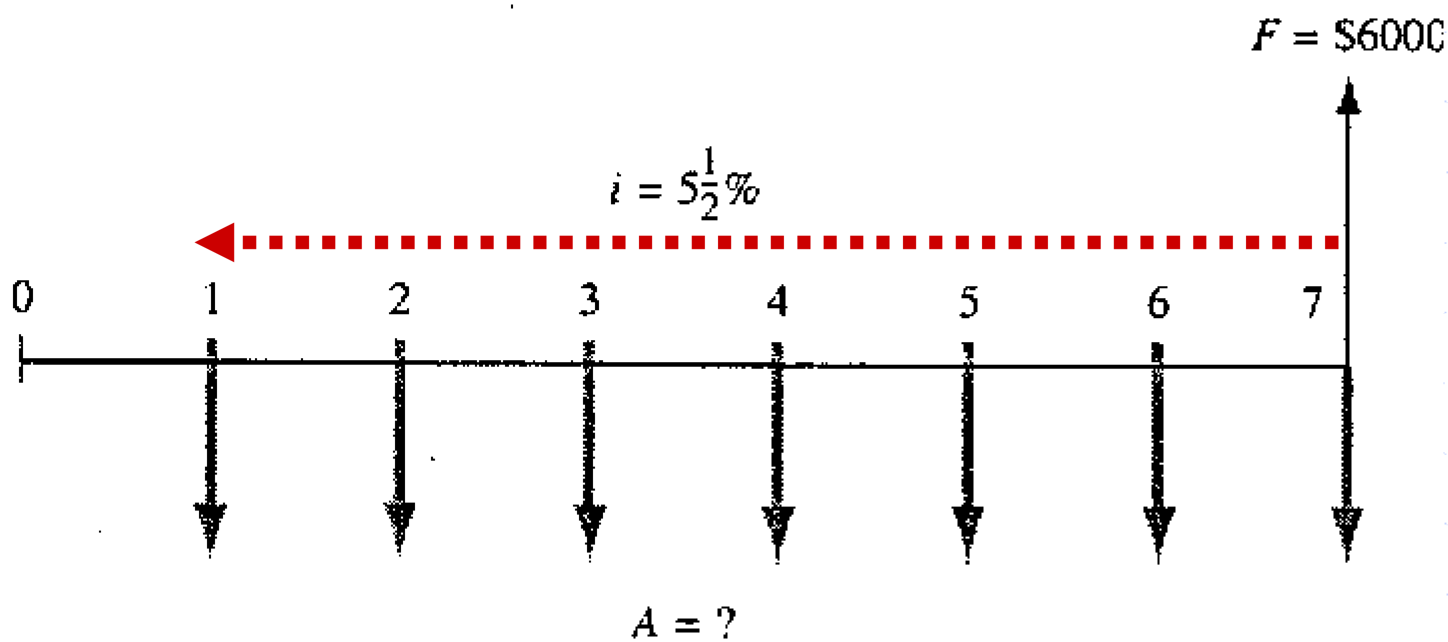
- **Simplifying, we get:**

$$A = F\left[\frac{i}{(1+i)^n - 1}\right]$$

  - **The term in the brackets is (A/F, i%, n)**
  - **Sinking fund factor (from the year 1724!)**

Example 2.6 (from the book)

- How much money must Carol save each year (starting 1 year from now) at 5.5%/year:
  - In order to have $6000 in 7 years?



Example 2.6 (continued)

- **Solution:**
  - The cash flow diagram fits the A/F factor (future amount given, annual amount??)
  - A= \$6000 (A/F, 5.5%, 7) = 6000 (0.12096) = <u>\$725.76</u> per year
  - The value 0.12096 can be computed (using the A/F formula), or looked up in a table

Converting from Annual to Future

- **Given**

$$A = F\left[\frac{i}{(1+i)^n - 1}\right]$$

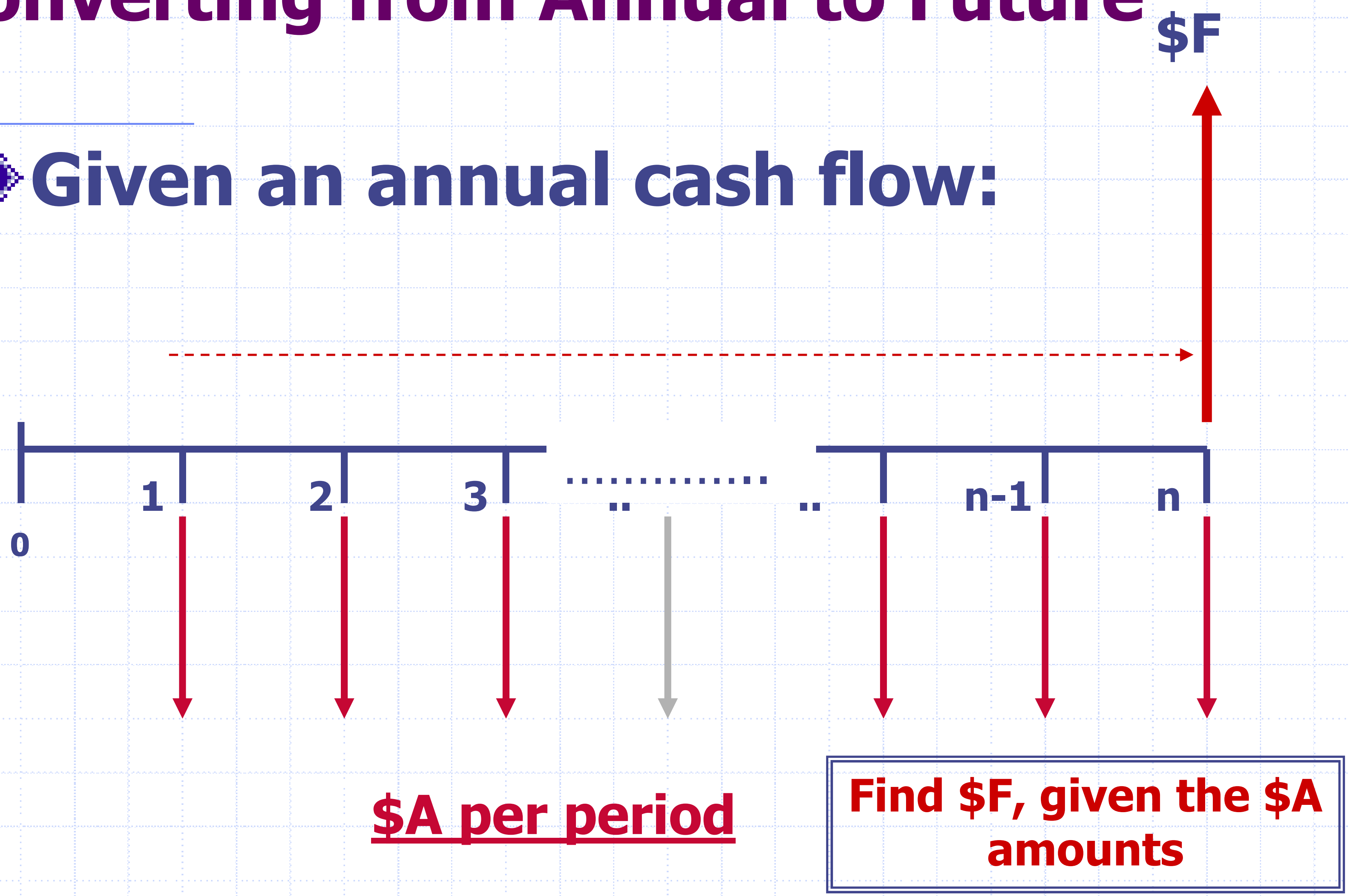
- **Solve for F in terms of A:**

$$F = A\left[\frac{(1+i)^n - 1}{i}\right]$$

  - **The term in the brackets is (F/A, i%, n)**
  - **Uniform series compound amount factor**

Converting from Annual to Future

- Given an annual cash flow:



More Numerical Examples

How Fast Does Our Money Grow?

- Invest $1000 now for 64 years at 6%:
  - $F = P(1+i)^n = \$1000\,(1.06)^{64} = \$41{,}647$
    - Things get *big* over time!
- Invest $1000 each year for 64 years at 6%:
  - $F = A\,[(1+i)^n - 1]/i$
    - $= \$1000\,[(1.06)^{64} - 1]/.06 = \$677{,}450$
    - This is *really* big!

Non-Equal, Non-Annual Payments

- **Same basic ideas still work**
- **Assume that you plan to invest:**
  - \$2000 in year 0
  - \$1500 in year 2
  - \$1000 in year 4
- **How much will you have in year 10?**
  - $\$2000\,(1+i)^{10} + \$1500\,(1+i)^{8} + \$1000\,(1+i)^{6}$

A More Complicated Example

- **How much to invest (at 5%) to get:**
  - \$1200 in year 5
  - \$1200 in year 10
  - \$1200 in year 15
  - \$1200 in year 20
- **Convert each future amount to present:**
  - According to $P = F/(1+i)^n$
  - Invest $\$1200/(1.05)^5 + \$1200/(1.05)^{10} + \$1200/(1.05)^{15} + \$1200/(1.05)^{20} = \$2706$

Sensitivity Analysis (Section 2.9)

Sensitivity Analysis

- So far, we have assumed that all of the numbers and parameters are known with certainty:
  - In reality, most of them will be **estimates!**
- We can use **sensitivity analysis** to assess the impact of each input parameter on the output variable of interest (present worth, internal rate of return, etc.):
  - Best performed using a **spreadsheet!**
  - Vary the input parameters within ranges, observe the change in the output variable

Sensitivity Analysis

- Perform "what-if" analysis on one or more input parameters:
  - Observe any changes in the output variable
- You can easily do this by hand in a spreadsheet
- Commercial Excel add-ins are also available:
  - For example, Palisade Corporation's ***TopRank***

Sensitivity Analysis

- Varying one or more input parameters:
  - Store the results of each change
  - Plot the results as a function of input values
- If a small change in an input parameter leads to a large change in the output:
  - Then the output is "**sensitive**" to that input
- Either more effort should go into estimating that parameter:
  - Or you should choose a decision that is **not** sensitive to that input!

Sensitivity Analysis

- If the output is not as sensitive to some input parameters:
  - Then not as much effort is required to estimate those parameters!
  - Because they do not have much impact on the output variable of interest

Sensitivity Analysis

- You may see some sensitivity analysis on the homework assignments:
  - We will discuss this more in Chapter 18

Summary

Summary

- This lecture presented the fundamental **time-value-of-money relationships** common to most engineering economic analysis calculations
- We learned how to convert:
  - Present to future, and vice versa
  - Annual to present, and vice versa
  - Future to annual, and vice versa
- We saw that costs get ***big*** over time
- We learned about **sensitivity analysis**

Summary

- You **must** master these basic relationships in order to use them in economic analysis and decision making:
  - These relationships **will be important to you**, both professionally and **in your personal life!**
  - Make sure you have a good grasp of these concepts, **so you can use them correctly!**