Hebron University Research Journal (A).
Vol.(5), pp.(93 – 99), 2011

H.U.R.J. is available online at
http://www.hebron.edu/journal

The Spectrum of the Coloration Matrix for the Complete Partite Graph

Subhi Ruzieh

Al-Najah National University

ABSTRACT:

This paper talks about eigenvalues of graphs and finding them through making use of the coloration partition. We first define the concept of coloration. We then apply it to compute eigenvalues of some graphs. The concentration will be on graphs having a nontrivial coloration like the complete bipartite graphs K(m, n) and the complete 3 - partite graphs K(m, n, p). The idea can be carried through to the complete n-partite graphs, but the computations then will be much more complicated.

الملخص:

يتناول البحث القيم المميزة للرسوم وايجادها عن طريق ما يمكن تسميته التجزئة التلوينية. نعرض اولا تعريفا لهذا المفهوم. ثم نطبقه على بعض الرسوم لحساب القيم المميزة. يتم التركيز على الرسوم التي لها تجزئة تلوينية غير التقليدية مثل الثنائي الكامل والثلاثي الكامل. يمكن تعميم المبدأ على الحالة العامة للرسوم الكاملة متعددة الاجزاء لكن الحسابات هناك تأخذ طابعا اكثر تعقيدا.

* **Corresponding author:** ruziehs@najah.edu

INTRODUCTION:

We first define the term coloration. This could be found in many references on graph theory. See for example [Ruzieh, 1993, pp 17-23]. In most of the work done before, the term coloration was strongly connected with vertex coloring in a graph. Bounds were found on the chromatic index of a graph in terms of the spectral radius of the adjacency matrix of the graph. Very little was done in the direction of eigenvalues as related to the concept of coloration. This work will utilize the concept of coloration to compute eigenvalues of some graph matrices.

Let G be a graph of order p having a vertex set V. Let the set V be partitioned into n classes $V_1, V_2, V_3, \ldots, V_n$. Define the indicator matrix $M = [m_{ij}]$ to be the pxn (0, 1) matrix whose i,j-entry is equal to one if vertex v_i belongs to class V_j and is zero otherwise. The coloration matrix B (if exists) is the nxn matrix that satisfies the relation QM = MB, where Q is a square graph matrix which could be the adjacency matrix A, the distance matrix D, or the matrix of reciprocals of the non-zero distances R. If Q = A then the i,j entry in the matrix $B = [b_{ij}]$, denoted by b_{ij}, is the number of vertices in class V_j that are adjacent to an arbitrary vertex in class V_j , i. e, any vertex in class V_i is adjacent to b_{ij} vertices in class V_j. It is worth noting that the number b_{ij} depends only on the choice of the classes V_i and V_j and not on the choice of a particular vertex in the class. This holds for any graph. In the worst case, any graph has at least the trivial coloration where any coloration class contains just one vertex. In such a case, the coloration matrix B is the same as the adjacency matrix of the graph.

We may consider as an example the complete bipartite graph $K_{4,5}$. The partite classes are V_1 and V_2 with sizes 4 and 5 respectively. Every vertex in the first class is adjacent to zero vertices in the first class and to 5 vertices in the second class. Every vertex in the second class is adjacent to 4 vertices in the first class and to zero vertices in the second class. The coloration matrix is

$$B = \begin{pmatrix} 0 & 5 \\ 4 & 0 \end{pmatrix}$$

With this definition, we have the following easily seen properties:

1.the matrix B is a nonnegative integral matrix.

2.the sum of entries in row(i) of the matrix B is the degree of a vertex in class V_i and all vertices in any one coloration class have the same degree. This is true in general for any simple graph that has a non trivial coloration.

3.the matrix $S = M^tM$ is a diagonal matrix whose diagonal entries are the sizes of the classes. Thus $B = S^{-1}M^tAM$

4.if the matrix A is the adjacency matrix of an r-regular graph, then the matrix B has a constant row sum equal to r.

SECTION 1. COLORATION THEOREM

Concerning eigenvalues and characteristic polynomials, we have the following simple but important theorem.

THEOREM (1) «SEE THEOREM 4.7 CVETKOVIC , ET AL., 1980, P 123»

The characteristic polynomial PB(t) of the matrix B divides the characteristic polynomial PA(t) of the matrix A

Proof.

The proof may be seen in the reference stated. (A halmos)

Regarding this theorem we have the following properties; see Ruzieh, 1989, pp 21-22.

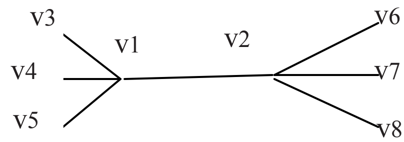
1. If λ is a nonzero eigenvalue of B then the eigenvector entries of the vector Mz are constant on any given class Vi, i.e, if vi, vj Vk then (Mz)i = (Mz)j

2.If λ is a nonzero eigenvalue of A but not of B then the eigenvector entries of Mz add up to zero on any class Vi .

If x is a non zero eigenvector of A corresponding to the eigenvalue λ, then Ax = λ x which gives MtAt x = λ Mt x. This gives (AM)tx = λ Mt x which in turn gives (MB)t x = λ Mt x. Consequently we get BtMt x = λ Mt x. If x is a nonzero vector, then λ is an eigenvalue of B in which case the entries of the vector x are constant over vertices in the same class. If the vector Mtx is the zero vector, then the sum of the vector x entries over any class is zero. In this case the numbe λ is not an eigenvalue of B.

This last case does indeed happen. In some cases, the matrix B captures all of the nonzero eigenvalues of the matrix A, as in the case of the complete bipartite graph. Yet there are cases where the matrix B doesn't capture all of the nonzero eigenvalues of the matrix A. In what follows we present an example of a graph where A is the adjacency matrix of the graph and B is its coloration matrix.

The graph G:



The coloration classes are

V1 = {v1, v2 }, V2 = {v3, v4, v5, v6, v7, v8 }

The coloration matrix is

$$B = \begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix}$$

The adjacency matrix A has the following spectrum

σ (A) = { 2.302776, 1.302776, 0, 0, 0, 0, -1.302776, -2.302776 }

While the matrix B has the following spectrum

σ(B)= { 2.302776, -1.302776 }.

In the case of the complete bipartite graph of order n, it is true that $P_A(\lambda) = \lambda^{n-2} P_B(\lambda)$ but this is not true in general.

3. The spectral radius of A is also the spectral radius of B..

4. If A is the adjacency matrix of a distance regular graph then the matrix B captures all of the nonzero eigenvalues of the matrix A.

Note: If the matrix Q is the distance matrix of the graph, then the entry bij in the matrix B will refer to the sum of distances of a vertex in class Vi to the vertices in class Vj. While if the matrix Q is the matrix of the reciprocals of the non zero distances between the vertices, then the entry bij in the matrix B refers to the sum of the reciprocals of distances of a vertex in class Vi to the vertices in class Vj . When talking about distances, it is assumed that the graph is connected.

We restrict our computations to the complete n-partite graph, for n = 2 and 3.In this case, there will be 2 or 3 partite classes respectively.

SECTION 2. THE COMPLETE BIPARTITE GRAPH

We consider first the graph K(m, n). The matrices mentioned at the end of the last section related to this graph will be stated below. In the notation used, the symbol J_{mn} refers to the $m_x n$ matrix of all ones and O_{MN} refers to the mxn matrix of all zeros and I_n refers to the $n_x n$ identity matrix. The matrices of concern in the block notation are:

The adjacency matrix

$$A = \left(\begin{array}{c|c} O_{mm} & J_{mn} \\ \hline J_{nm} & O_{nn} \end{array}\right)$$

The distance matrix

$$D = \left(\begin{array}{c|c} 2(J-I)_{mm} & J_{mn} \\ \hline J_{nm} & 2(J-I)_{nn} \end{array}\right)$$

Each of the above matrices has the form

$$T(c) = \left(\begin{array}{c|c} c(J-I)_{mm} & J_{mn} \\ \hline J_{nm} & c(J-I)_{nn} \end{array}\right)$$

where c takes the values 0, 2 and .5 for the matrices A, D and R, respectively. We define the matrix

$H(c) = T(c) + c\, I_{m+n}$. The matrix H is explicitly the matrix 6 bage49:

$$H(c) = \left(\begin{array}{c|c} (cJ)_{mm} & J_{mn} \\ \hline J_{nm} & (cJ)_{nn} \end{array}\right)$$

In the previous notation, we have QB = MB. We are back to the same relation but for the matrix Q = T(c) = H(c) - cI. The matrix H has clearly two independent rows. It thus has rank equal to 2 and (m + n - 2)) zero eigenvalues. The two nonzero eigenvalues form the spectrum of the matrix B where HM = MB. The 2x2 diagonal matrix S = M^tM containing the sizes of the partition classes V_1 and V_2 is then

$$S = \begin{pmatrix} 0 & m \\ n & 0 \end{pmatrix}$$

The matrix B, being a coloration matrix, is the matrix $B = S^{-1}M^tHM$ and simple calculations show that

$$B = \begin{pmatrix} n & cm \\ cn & m \end{pmatrix}$$

The characteristic polynomial of the matrix B is

$$P_B(t) = (t - cm)(t - cm) - mn$$
$$= t^2 - c(m + n)t + mnc^2 - mn$$

and the eigenvalues of the matrix B are

H(c) =

$$
\left(\begin{array}{cccc|cccc}
c & \cdots & c & c & 1 & \cdots & 1 & 1 \\
c & \cdots & c & c & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
c & \cdots & c & c & 1 & \cdots & 1 & 1 \\
\hline
1 & \cdots & 1 & 1 & c & \cdots & c & c \\
1 & \cdots & 1 & 1 & c & \cdots & c & c \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & \cdots & 1 & 1 & c & \cdots & c & c
\end{array}\right)
$$

MATRIX 6

$\lambda 1, \lambda 2 =$

$$\frac{cm + cn \pm \sqrt{c\hat{\ }2\,(m-n)^2 + 4mn}}{2}$$

So the spectrum σ(H) of the matrix H(c) is
σ(H(c)) = {λ1, λ2, 0, …, 0} with zero having multiplicity m+n–2. The spectrum of the matrix T is then
σ(T(c)) = { λ1 – c, λ_2 – c, - c, -c, …, -c} where -c has multiplicity (m+n-2)
Next we state the theorem that gives explicitly the spectra for the adjacency matrix A, the distance matrix D and the matrix R. The values of the constant c corresponding to these matrices are c = 0, 2 and .5 respectively. Using the previous results we get the following theorem.

THEOREM (2)

For the complete bipartite graph K(m, n) we have the following results.

(1) The spectrum of the adjacency matrix A is : σ(A) =

{ $\sqrt{mn}$, $\sqrt{m\ n}$, 0, 0, …, 0}
where zero has multiplicity (m + n – 2)

(2) The spectrum of the distance matrix D is

σ(D) ={m + n – 2 $\pm$ $\sqrt{m^2 + n^2 - m}$, -2, …, -2} where -2 has multiplicity (m + n –2)

(3)The spectrum of the matrix R is
σ(R) =

{ $\dfrac{m + n - 2 \pm \sqrt{m^2 + n^2 + 14mn}}{4}$

,-1/2,...,-1/2 } where - ½ has multiplicity m + n – 2.

SECTION 3. THE COMPLETE 3 - PARTITE GRAPH

Consider the complete 3 – Partite Graph K(m, n, k)
The matrix H(c) as defined earlier has the following partitioned form

$$H(c) = \left(\begin{array}{c|c|c} (cJ)_{mm} & J_{mn} & J_{mk} \\ \hline J_{nm} & (cJ)_{nn} & J_{nk} \\ \hline J_{km} & J_{kn} & (cJ)_{kk} \end{array}\right)$$

$$\text{Then } T(c) = \begin{cases} A & \text{if } c = 0 \\ D & \text{if } c = 2 \\ ½ = R & \text{if } c \end{cases}$$

The indicator matrix, M, is an (m + n + k)x3, (0, 1) matrix and the matrix S, is the following (3x3) matrix

$$S = \begin{pmatrix} 0 & 0 & m \\ 0 & n & 0 \\ k & 0 & 0 \end{pmatrix}$$

where m. n and k are the sizes of the three partite classes respectively.
The general coloration matrix $B = S^{-1}M^tHM$ is the following (3, 3) matrix

$$B = \begin{pmatrix} k & n & cm \\ k & cn & m \\ ck & n & m \end{pmatrix}$$

The characteristic polynomial of the matrix B is
$P_B(t) = t^3 - c(m + n + k)\, t^2 + (c^2 - 1)(mn + mk + nk)\, t - mnk(c^3 - 3c + 2)$
The matrix H has clearly (n + m + k – 3) zero eigenvalues and just three nonzero eigenvalues whose product is
$\lambda_1\lambda_2\lambda_3 = mnk(c^3 - 3c + 2)$ and whose sum is
$\lambda_1 + \lambda_2 + \lambda_3 = c(m + n + k)$

The following theorem gives the spectra of the adjacency matrix A, the distance matrix D and the matrix R for the graph K(m, n, k)

THEOREM (3):

For the graph K(m, n, k), the spectra of the matrices A, D and R, are as follows:
(1)By letting c = 0 , the spectrum of the adjacency matrix contains (m + n + k – 3) zero eigenvalues and the other three satisfy the equation
$\lambda 3 - (mn + mk + nk) \lambda - 2mnk = 0$

(2)The spectrum of the distance matrix D contains -2 as an eigenvalue with multiplicity (m+n+k – 3) and three simple eigenvalues $\mu_i = \lambda_i - 2$ for i = 1, 2 and 3 where λ_i satisfy the equation
$\lambda^3 - 2(m + n + k) \lambda^2 + 3(mn + mk + nk) \lambda - 4mnk = 0$

(3)Letting c = .5 the spectrum of the matrix R contains -.5 as an eigenvalue with multiplicity (m + n + k – 3) and three simple eigenvalues $\alpha_i = \lambda i - .5$ for i = 1, 2 and 3 where λ_i satisfies the equation
$\lambda 3 - .5(m + n + k) \lambda 2 - .75 (mn + mk + nk) \lambda - .625 mnk = 0$
In the case of the complete n-partite graph, the matrix B will be an (n, n) matrix and its eigenvalues will be the roots of an n^{th} degree polynomial and the number of zero eigenvalues in this case is (p – n) where p denotes the number of vertices (i.e order of the graph).

REFERENCES

1. Cvetkovic, D. Doob and Sachs, H. (1980): **Spectra of Graphs.** New York, Academic Press.

2. Ruzieh, S. (1993). " Coloration matrices". **Abhath Al- Yarmouk.** Vol2, no 2, 1993, pp. 17-23

3. Ruzieh, S. (1989). "Some Applications of Matrices Related to Graphs". Unpublished Ph. D thesis. Math and Computer Science Department. Clarkson University. Potsdam. N.Y. U. S. A