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SPIN AND PSEUDOSPIN SYMMETRIES IN RELATIVISTIC TRIGONOMETRIC PÖSCHL–TELLER POTENTIAL WITH CENTRIFUGAL BARRIER

M. HAMZAVI*

Department of Basic Sciences, Shahrood Branch,
Islamic Azad University, Shahrood, Iran
majid.hamzavi@gmail.com

S. M. IKHDAIR

Physics Department, Near East University,
922022 Nicosia, North Cyprus, Mersin 10, Turkey
sikhdair@neu.edu.tr

K.-E. THYLWE

KTH-Mechanics, Royal Institute of Technology,
S-100 44 Stockholm, Sweden
ket@mech.kth.se

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Approximate analytical solutions of the Dirac equation with the trigonometric Pöschl–Teller (tPT) potential are obtained for arbitrary spin-orbit quantum number κ using an approximation scheme to deal with the spin-orbit coupling terms $\kappa(\kappa \pm 1)r^{-2}$. In the presence of exact spin and pseudo-spin (p-spin) symmetric limitation, the bound state energy eigenvalues and the corresponding two-component wave functions of the Dirac particle moving in the field of attractive and repulsive tPT potential are obtained using the parametric generalization of the Nikiforov–Uvarov (NU) method. The case of nonrelativistic limit is studied too.

Keywords: Dirac equation; trigonometric Pöschl–Teller (tPT) potential; spin and p-spin symmetry; NU method; approximation schemes.

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*Corresponding author.

1. Introduction

It is well known that the exact energy eigenvalues of the bound states play an important role in quantum mechanics. In particular, the Dirac equation, which describes the motion of a spin-1/2 particle, has been used in solving many problems of nuclear and high-energy physics. The spin and the p-spin symmetries of the Dirac Hamiltonian had been discovered many years ago; however, these symmetries have recently been recognized empirically in nuclear and hadronic spectroscopes.[1] Within the framework of Dirac equation, p-spin symmetry is used to feature the deformed nuclei and the super deformation to establish an effective shell model,[2–4] whereas spin symmetry is relevant for mesons.[5] The spin symmetry occurs when the scalar potential $S(r)$ is nearly equal to the vector potential $V(r)$ or equivalently $S(r) \approx V(r)$ and p-spin symmetry occurs when $S(r) \approx -V(r)$.[6,7] The p-spin symmetry refers to a quasi-degeneracy of single nucleon doublets with nonrelativistic quantum number $(n, l,\ j = l + 1/2)$ and $(n-1,\ l+2,\ j = l+3/2)$, where n, l and j are single nucleon radial, orbital and total angular quantum numbers, respectively.[8,9] The total angular momentum is given by $j = \tilde{l} + \tilde{s}$, where $\tilde{l} = l + 1$ pseudo-angular momentum and $\tilde{s}$ is p-spin angular momentum.[10,11] Liang *et al.*[12] investigated the symmetries of the Dirac Hamiltonian and their breaking in realistic nuclei in the framework of perturbation theory. Guo[13] used the similarity renormalization group to transform the spherical Dirac operator into a diagonal form and then the upper (lower) diagonal element became an operator describing Dirac (anti-)particle, which holds the form of the Schrödinger-like operator with the singularity disappearing in every component. Chen and Guo[14] investigated the evolution toward the nonrelativistic limit from the solutions of the Dirac equation by a continuous transformation of the Compton wavelength λ. Lu *et al.*[15] recently showed that the p-spin symmetry in single particle resonant states in nuclei is conserved when the attractive scalar and repulsive vector potentials have the same magnitude but opposite sign.

The tPT potential has been proposed for the first time by Pöschl and Teller[16] in 1933 to describe the diatomic molecular vibration. Chen[17] and Zhang *et al.*[18] have studied the relativistic bound state solutions for the tPT potential and hyperbolical PT (Second PT) potential, respectively. Liu *et al.*[19] studied the tPT potential within the framework of the Dirac theory. Recently, Candemir[20] investigated the analytical $\tilde{s}$-wave solutions of Dirac equation for tPT potential under the p-spin symmetry condition. Very recently, Hamzavi and Rajabi[21] studied the exact s-wave solution ($l = 0$) of the Schrödinger equation for the vibrational tPT potential. The tPT takes the form:

$$V_{\rm tPT}(r) = \frac{V_1}{\sin^2(\alpha r)} + \frac{V_2}{\cos^2(\alpha r)}, \quad 0 < \alpha r < \pi/2\,, \tag{1}$$

where the parameters V_1 and V_2 describe the property of the potential while the parameter α is related to the range of this potential.[19] We find out that this potential has a minimum value at $r_0 = \frac{1}{\alpha}\tan^{-1}\left(\sqrt[4]{\frac{V_1}{V_2}}\right)$. For the case when $V_1 = V_2$, the minimum value is at $r_0 = \frac{\pi}{4\alpha} \in (0, \infty)$ for $\alpha > 0$. The second derivative which

determines the force constants at $r = r_0$ is given by

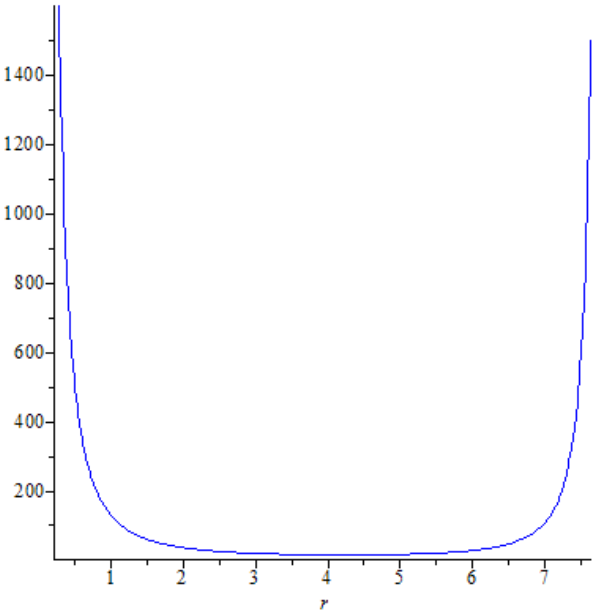
$$\left.\frac{d^2V}{dr^2}\right|_{r=r_0} = \frac{8\alpha^2(V_2 + \sqrt{V_1V_2})}{\cos^2\left[\tan^{-1}\left(\sqrt[4]{\frac{V_1}{V_2}}\right)\right]}, \tag{2}$$

for any α value and thus

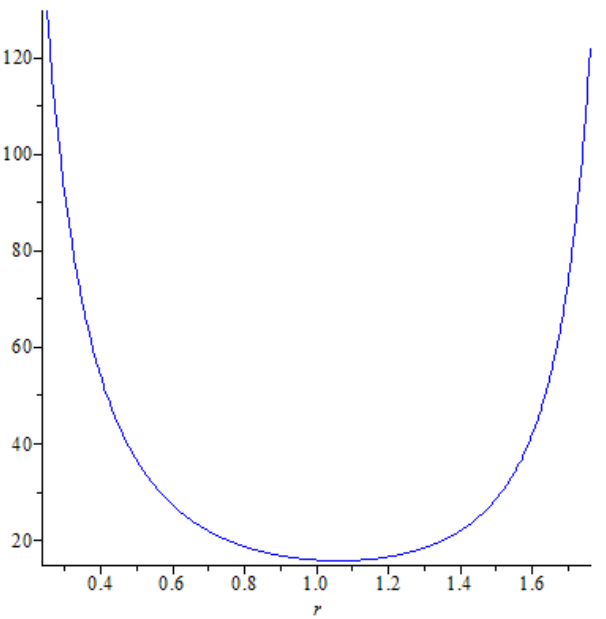
$$V(r_0) = \frac{\sqrt{V_1V_2} + V_2}{\cos^2\left[\tan^{-1}\left(\sqrt[4]{\frac{V_1}{V_2}}\right)\right]}, \tag{3}$$

which means that $V(r)$ at $r = r_0$ has a relative minimum for $\alpha > 0$. When $V_1 = V_2 = V$ then minimum value is $V(r_0) = 4V$ and $\frac{d^2V}{dr^2}|_{r=r_0} = 32\alpha^2V$. In Figs. 1(a) and 1(b), we draw the tPT potential (1) for parameter values $V_1 = 5.0$ fm^{-1}, $V_2 = 3.0$ fm^{-1}, $\alpha = 0.02$ fm^{-1} and $\alpha = 0.30$ fm^{-1}. Here the potential has a minimum value at $r_0 = 0.27027\pi/\alpha$. The curve is nodeless in $\alpha r \in (0, \pi/2)$. For example, with $\alpha = 0.30$ fm^{-1}, $r_0 = 2.8303$ fm and minimum potential $V(r_0 = 2.8303 \text{ fm}) = 15.746$ fm^{-1}. It is worthy to note that in the limiting case when $\alpha \to 0$, the tPT potential can be reduced to the Kratzer potential[22,23] $V(r) = D_e(\frac{r-r_e}{r})^2 + \eta$, where r_e is the equilibrium intermolecular separation and D_e is the dissociation energy between diatomic molecules. In our case, $D_e = V_1$, $\eta = V_2$ and $r_e = 1/\alpha$. In the case of $\eta = 0$, it reduces to the molecular potential which is called the modified Kratzer potential proposed by Simons *et al.*[24] and Molski and Konarski.[25] In the case of $\eta = -D_e$, this potential turns into the Kratzer potential, which includes an attractive Coulomb potential and a repulsive inverse square potential introduced by Kratzer in 1920.[26] The aim of this present work is to extend our previous work[20] to the relativistic case and $\kappa \neq \pm 1$ (rotational case). We introduce a convenient approximation scheme to deal with the strong singular centrifugal term. The ansatz of this approximation possesses the same form of the potential and is singular at the origin as the centrifugal term r^{-2}.

We want to solve the Dirac equation with flexible parameters tPT potential model. However, the Dirac-tPT problem can no longer be solved $\kappa \neq \pm 1$ in a closed form due to the existence of spin-orbit coupling term $\kappa(\kappa \pm 1)r^{-2}$ and it is necessary to resort to approximation methods. Therefore, we use an approximation scheme to deal with this term and solve approximately the Dirac equation with the tPT potential for arbitrary spin-orbit quantum number κ. In the presence of spin and p-spin symmetric limitation, we obtain the approximate relativistic bound state solutions including the energy eigenvalue equations and the corresponding unnormalized upper- and lower-spinor components of the wave functions using the concepts of parametric generalization of the NU method,[27] since the relativistic corrections are not neglected.



(a)



(b)

Fig. 1. (a) A plot of the tPT potential for $\alpha = 0.2$ fm^{-1}. (b) A plot of the tPT potential for $\alpha = 0.8$ fm^{-1}.

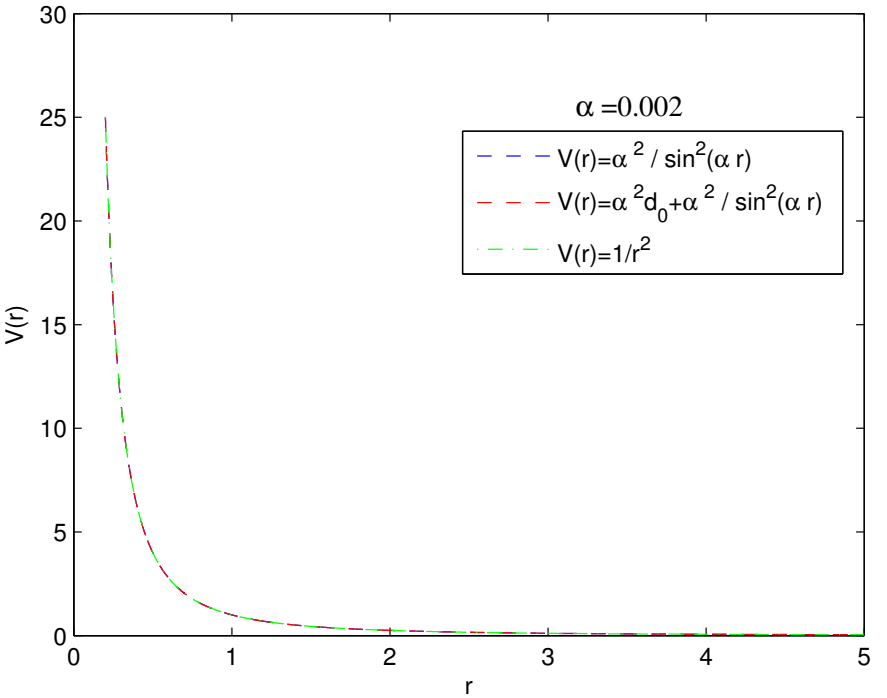


Fig. 2. (Color online) The centrifugal term $1/r^2$ (green line) and its approximations (Eq. 8).

Over the past years, the Nikiforov–Uvarov (NU) method[28] has shown to be a powerful tool in solving the second-order differential equation. It was applied successfully to a large number of potential models.[29–35] This method has also been used to solve the spinless (spin-0) Schrödinger[36–40] and Klein–Gordon (KG)[41–46] equations and also relativistic spin-1/2 Dirac equation[47–52] with different potential models.

The structure of the paper is as follows. In Sec. 2, in the context of spin and p-spin symmetry, we briefly introduce the Dirac equation with scalar and vector tPT potentials for arbitrary spin-orbit quantum number κ. The parametric generalization of the NU method is displayed in Appendix A. In the presence of the spin and p-spin symmetry, the approximate energy eigenvalue equations and corresponding two-component wave functions of the Dirac-tPT problem are obtained. The nonrelativistic limit of the problem is discussed in this section too. Finally, our final concluding remarks are given in Sec. 3.

2. Bound State Solutions

The Dirac equation for fermionic massive spin-1/2 particles moving in the field of an attractive scalar $S(r)$ and a repulsive vector $V(r)$ potential (in units $\hbar = c = 1$)

is

$$[\boldsymbol{\alpha}\cdot\mathbf{p}+\beta(M+S(r))]\psi(\mathbf{r})=[E-V(r)]\psi(\mathbf{r})\,, \tag{4}$$

where E is the relativistic energy of the system, $\mathbf{p}=-i\boldsymbol{\nabla}$ is the three-dimensional (3D) momentum operator and M is the mass of the fermionic particle. $\boldsymbol{\alpha}$ and β are the 4×4 usual Dirac matrices.[53] One may closely follow the procedure described in Eqs. (17)–(19) of Ref. 54 to obtain,

$$\left[\frac{d^2}{dr^2}-\frac{\kappa(\kappa+1)}{r^2}\right]F_{n\kappa}(r)+\frac{\dfrac{d\Delta(r)}{dr}}{M+E_{n\kappa}-\Delta(r)}\left(\frac{d}{dr}+\frac{\kappa}{r}\right)F_{n\kappa}(r)$$
$$=[(M+E_{n\kappa}-\Delta(r))(M-E_{n\kappa}+\Sigma(r))]F_{n\kappa}(r),\quad r\in(0,\infty)\,, \tag{5}$$

$$\left[\frac{d^2}{dr^2}-\frac{\kappa(\kappa-1)}{r^2}\right]G_{n\kappa}(r)+\frac{\dfrac{d\Sigma(r)}{dr}}{M-E_{n\kappa}+\Sigma(r)}\left(\frac{d}{dr}-\frac{\kappa}{r}\right)G_{n\kappa}(r)$$
$$=[(M+E_{n\kappa}-\Delta(r))(M-E_{n\kappa}+\Sigma(r))]G_{n\kappa}(r),\quad r\in(0,\infty)\,, \tag{6}$$

where $\kappa(\kappa-1)=\tilde{l}(\tilde{l}+1)$ and $\kappa(\kappa+1)=l(l+1)$. The spin-orbit quantum number κ is related to the orbital quantum numbers l and $\tilde{l}$ for spin symmetry and p-spin symmetric models, respectively, as

$$\kappa=\begin{cases}-(l+1)=-\left(j+\dfrac{1}{2}\right)(s_{1/2},p_{3/2},\ldots), & j=l+\dfrac{1}{2},\ \text{aligned spin}\ (\kappa<0)\,,\\[2ex] +l=+\left(j+\dfrac{1}{2}\right)(p_{1/2},d_{3/2},\ldots), & j=l-\dfrac{1}{2},\ \text{unaligned spin}\ (\kappa>0)\,.\end{cases}$$

Further, κ in the quasi-degenerate doublet structure can be expressed in terms of $\tilde{s}=1/2$ and $\tilde{l}$, the p-spin and pseudo-orbital angular momentum, respectively, as

$$\kappa=\begin{cases}-\tilde{l}=-\left(j+\dfrac{1}{2}\right)(s_{1/2},p_{3/2},\ldots), & j=\tilde{l}-\dfrac{1}{2},\ \text{aligned p-spin}\ (\kappa<0),\\[2ex] +(\tilde{l}+1)=+\left(j+\dfrac{1}{2}\right)(d_{3/2},f_{5/2},\ldots), & j=\tilde{l}+\dfrac{1}{2},\ \text{unaligned p-spin}\ (\kappa>0),\end{cases}$$

where $\kappa=\pm1,\pm2,\ldots$. For example, the states $(1s_{1/2},0d_{3/2})$ and $(1p_{3/2},0f_{5/2})$ can be considered as p-spin doublets.

2.1. *Spin symmetric limit*

In the spin symmetric limitation, $\frac{d\Delta(r)}{dr}=0$ or $\Delta(r)=C_s=$ constant,[7,55–58] then Eq. (5) with $\Sigma(r)=V_{\rm tPT}(r)$, becomes

$$\left[\frac{d^2}{dr^2}-\frac{\kappa(\kappa+1)}{r^2}-\gamma\left(\frac{V_1}{\sin^2(\alpha r)}+\frac{V_2}{\cos^2(\alpha r)}\right)-\beta^2\right]F_{n\kappa}(r)=0\,, \tag{7a}$$

$$\gamma=M+E_{n\kappa}-C_s\ \ \text{and}\ \ \beta^2=(M-E_{n\kappa})(M+E_{n\kappa}-C_s)\,, \tag{7b}$$

where $\kappa = l$ and $\kappa = -l-1$ for $\kappa < 0$ and $\kappa > 0$, respectively. The Schrödinger-like equation (7a) that results from the Dirac equation is a second-order differential equation containing a spin-orbit centrifugal term $\kappa(\kappa+1)r^{-2}$ which has a strong singularity at $r = 0$, and needs to be treated very carefully while performing the approximation. Equation (7a) has an exact rigorous solution only for the states with $\kappa = -1$ because of the existence of the centrifugal term $\kappa(\kappa+1)/r^2$. However, when this term is taken into account, the corresponding radial Dirac equation can no longer be solved in a closed form and it is necessary to resort to approximate methods. Over the last few decades several schemes have been used to calculate the energy spectrum. The main idea of these schemes relies on using different approximations of the spin-orbit centrifugal coupling term $\kappa(\kappa+1)/r^2$. So we need to perform a new approximation for the spin-orbit term as a function of the tPT potential parameters. Therefore, we resort to use an appropriate approximation scheme to deal with the centrifugal potential term as

$$\frac{1}{r^2} = \lim_{\alpha\to 0} \alpha^2 \left(d_0 + \frac{1}{\sin^2(\alpha r)} \right), \quad 0 < \alpha r \ll 1\,, \tag{8}$$

where $d_0 = 1/12$ is a dimensionless shifting parameter. The approximation (8) is done on the basis that $\sin(z) = z - z^3/3! + z^5/5! - z^7/7! + \cdots$, and in the limit when $z \to 0$, $\sin(z) \approx z$. To show the validity and accuracy of our choice to the approximation scheme (8), we plot the centrifugal potential term $1/r^2$ and its approximations: $\alpha^2/\sin^2(\alpha r)$ and $\alpha^2(d_0 + 1/\sin^2(\alpha r \sin^2(\alpha r))$ in Fig. 2. As illustrated, the three curves coincide together and show how accurate is this replacement. One of us has treated this problem in his recent work (see Ref. 59).

Thus, employing such an approximation scheme, we can then write Eq. (7a) as

$$\left[\frac{d^2}{dr^2} - \kappa(\kappa+1)\alpha^2 \left(d_0 + \frac{1}{\sin^2(\alpha r)} \right) - \gamma \left(\frac{V_1}{\sin^2(\alpha r)} + \frac{V_2}{\cos^2(\alpha r)} \right) - \beta^2 \right] \times F_{n\kappa}(r) = 0\,. \tag{9}$$

Followed by making a new change of variables $s(r) = \sin^2(\alpha r)$, this allows us to decompose the spin-symmetric Dirac equation (9) into the Schrödinger-type equation satisfying the upper-spinor component $F_{n,\kappa}(s)$,

$$\left\{ \frac{d^2}{ds^2} + \frac{\frac{1}{2} - s}{s(1-s)} \frac{d}{ds} - \frac{1}{s^2(1-s)^2} [-As^2 + Bs - C] \right\} F_{n,\kappa}(s) = 0\,,$$

$$A = -\frac{1}{4\alpha^2} [\kappa(\kappa+1)\alpha^2 d_0 + \beta^2]\,, \tag{10}$$

$$B = \frac{1}{4\alpha^2} [\kappa(\kappa+1)\alpha^2(1-d_0) + \gamma(V_1 - V_2) - \beta^2]\,,$$

$$C = \frac{1}{4\alpha^2} [\kappa(\kappa+1)\alpha^2 + \gamma V_1]\,,$$

Table 1. The specific values of the parametric constants for the spin symmetric Dirac-tPT problem.

Constant	Analytic value
c_4	$\frac{1}{4}$
c_5	$-\frac{1}{2}$
c_6	$\frac{1}{4}(1+4A)$
c_7	$-\frac{1}{4}(1+4B)$
c_8	$\frac{1}{16}(1+16C)$
c_9	$A-B+C+\frac{1}{16}$
c_{10}	$\frac{1}{2}\sqrt{1+16C}$
c_{11}	$2\sqrt{A-B+C+\frac{1}{16}}$
c_{12}	$\frac{1}{4}(1+\sqrt{1+16C})$
c_{13}	$\frac{1}{4}+\sqrt{A-B+C+\frac{1}{16}}$

where $F_{n\kappa}(r) \equiv F_{n,\kappa}(s)$ has been used. If the above equation is compared with (A.2), we can obtain the specific values for constants c_i $(i = 1, 2, 3)$ as

$$c_1 = \frac{1}{2}, \quad c_2 = 1 \ \text{ and } \ c_3 = 1\,.$$

In order to obtain the bound state solutions of Eq. (9), it is necessary to calculate the remaining parametric constants, i.e., c_i $(i = 4, 5, \ldots, 13)$ by means of the relation (A.5). Their specific values are displayed in Table 1 for the relativistic tPT potential model. Further, using these constants along with (A.10), we can readily obtain the energy eigenvalue equation for the Dirac-tPT problem as

$$\left(n+\frac{1}{2}+\sqrt{A-B+C+\frac{1}{16}}+\sqrt{C+\frac{1}{16}}\right)^2 = A\,, \tag{11}$$

or equivalently

$$\left(2n+1+\frac{1}{2}\sqrt{1+\frac{4V_2(M+E_{n\kappa}-C_s)}{\alpha^2}}+\frac{1}{2}\sqrt{(2\kappa+1)^2+\frac{4V_1(M+E_{n\kappa}-C_s)}{\alpha^2}}\right)^2$$

$$= \frac{1}{\alpha^2}(E_{n\kappa}-M)(M+E_{n\kappa}-C_s)-\kappa(\kappa+1)d_0\,. \tag{12}$$

To show the procedure of determining the energy eigenvalues from Eq. (12), we take a set of physical parameter values, $M = 10$ fm^{-1}, $V_1 = 5.0$ fm^{-1}, $V_2 = 3.0$ fm^{-1}, $C_s = 0$ fm^{-1} and $\alpha = 0.8, 0.6, 0.4, 0.2, 0.04, 0.02$.[19] In Tables 2 and 3, we present the energy spectrum for the spin symmetric case. Obviously, the pairs $(np_{1/2}, np_{3/2})$, $(nd_{3/2}, nd_{5/2})$, $(nf_{5/2}, nf_{7/2})$, $(ng_{7/2}, ng_{92})$, and so on are degenerate states. Thus, each pair is considered as spin doublet and has positive energy.[54] Further, when potential range parameter α approaches zero, the energy eigenvalues approaches a constant. From Eq. (12) we find that this constant is $M + V_1 + V_2 + 2\sqrt{V_1 V_2}$, i.e. $\lim_{\alpha\to 0} E_{n,\kappa} = M + V_1 + V_2 + 2\sqrt{V_1 V_2}$, where it can be seen from Tables 2 and 3, too. In addition, to show the accuracy of our approximation scheme, we have calculated the exact numerical energy eigenvalues of Eq. (7a) without making approximation to the centrifugal term by using the amplitude phase (AP) method.[60–63] As shown in Tables 2 and 3, our approximate energies obtained via the NU method is highly accurate (0.00012%) if compared with the exact ones obtained via AP in the low screening regime when the screening parameter values: $\alpha = 0.02$ fm^{-1} and 0.04 fm^{-1}. Also the approximate energies for the screening parameter values $\alpha = 0.2$ fm^{-1} and 0.4 fm^{-1} are also accurate (0.0078%). This means that the present approximation works well for the low values of α.

On the other hand, in order to establish the upper-spinor component of the wave functions $F_{n,\kappa}(r)$, namely, Eq. (7a), the relations (A.11)–(A.14) are used. First, we find the first part of the wave function as

$$\phi(s) = s^{\frac{1}{4}(1+\sqrt{1+16C})}(1-s)^{\frac{1}{4}+\sqrt{A-B+C+\frac{1}{16}}}\,. \tag{13}$$

Second, we calculate the weight function as

$$\rho(s) = s^{\frac{1}{2}\sqrt{1+16C}}(1-s)^{2\sqrt{A-B+C+\frac{1}{16}}}\,, \tag{14}$$

which gives the second part of the wave function as

$$y_n(s) = P_n^{(\frac{1}{2}\sqrt{1+16C}, 2\sqrt{A-B+C+\frac{1}{16}})}(1-2s)\,, \tag{15}$$

where $P_n^{(a,b)}(y)$ are the orthogonal Jacobi polynomials. Finally, the upper spinor component for arbitrary κ can be found through the relation (A.14)

$$F_{n\kappa}(s) = N_{n\kappa} s^{\frac{1}{4}(1+\sqrt{1+16C})}(1-s)^{\frac{1}{4}+\sqrt{A-B+C+\frac{1}{16}}} P_n^{(\frac{1}{2}\sqrt{1+16C}, 2\sqrt{A-B+C+\frac{1}{16}})}(1-2s) \tag{16}$$

or

$$F_{n\kappa}(r) = N_{n\kappa}(\sin(\alpha r))^{\frac{1}{2}(1+\eta_\kappa)}(\cos(\alpha r))^{\frac{1}{2}(1+\delta)} P_n^{(\frac{1}{2}\eta_\kappa, \frac{1}{2}\delta)}(\cos(2\alpha r))\,, \tag{17}$$

where

$$\eta_\kappa = \sqrt{1 + \frac{4}{\alpha^2}[\kappa(\kappa+1)\alpha^2 + \gamma V_1]}\,, \quad \delta = \sqrt{1 + \frac{4\gamma V_2}{\alpha^2}} \tag{18}$$

Table 2. The spin symmetric bound state energy eigenvalues (fm^{-1}) of the tPT potential for several values of n and κ with $M = 10.0$, $V_1 = 5.0$, $V_2 = 3.0$, $c_s = 0$ and $d_0 = 1/12$.

l	n, κ	$(l, j = l \pm 1/2)$	$\alpha = 0.8$ NU	$\alpha = 0.8$ AP	$\alpha = 0.4$ NU	$\alpha = 0.4$ AP	$\alpha = 0.2$ NU	$\alpha = 0.2$ AP	$\alpha = 0.04$ NU	$\alpha = 0.04$ AP	$\alpha = 0.02$ NU	$\alpha = 0.02$ AP
1	$1, -2, 1$	$1p_{3/2}, 1p_{1/2}$	29.02725	29.06111	27.36375	27.37213	26.54876	26.550841	25.90551	25.90559	25.82567	25.82570
2	$1, -3, 2$	$1d_{5/2}, 1d_{3/2}$	29.15386	29.39588	27.39629	27.45807	26.55699	26.57249	25.90584	25.90646	25.82575	25.82591
3	$1, -4, 3$	$1f_{7/2}, 1f_{5/2}$	29.34155	30.08052	27.44495	27.63985	26.56933	26.61874	25.90634	25.90833	25.82588	25.82638
4	$1, -5, 4$	$1g_{9/2}, 1g_{7/2}$	29.58785	31.17924	27.50956	27.94770	26.585773	26.69840	25.90701	25.91155	25.82605	25.82719
1	$2, -2, 1$	$2p_{3/2}, 2p_{1/2}$	31.16696	31.20131	28.43165	28.44013	27.08151	27.08361	26.01179	26.01187	25.87879	25.87881
2	$2, -3, 2$	$2d_{5/2}, 2d_{3/2}$	31.29020	31.53212	28.46376	28.52568	27.08969	27.10523	26.01212	26.01274	25.87887	25.87903
3	$2, -4, 3$	$2f_{7/2}, 2f_{5/2}$	31.47301	32.20983	28.51179	28.70668	27.10195	27.151391	26.01262	26.01460	25.878100	25.87950
4	$2, -5, 4$	$2g_{9/2}, 2g_{7/2}$	31.71308	33.30007	28.57555	29.01332	27.11828	27.230922	26.01328	26.01782	25.87916	25.88030

Table 3. The spin symmetric bound state energy eigenvalues (fm^{-1}) of the tPT potential for several values of n and κ with $d_0 = 0$.

l	n, κ	$(l, j = l \pm 1/2)$	$\alpha = 0.8$ NU	$\alpha = 0.8$ AP	$\alpha = 0.4$ NU	$\alpha = 0.4$ AP	$\alpha = 0.2$ NU	$\alpha = 0.2$ AP	$\alpha = 0.04$ NU	$\alpha = 0.04$ AP	$\alpha = 0.02$ NU	$\alpha = 0.02$ AP
1	$1, -2, 1$	$1p_{3/2}, 1p_{1/2}$	29.02464	29.06111	27.36305	27.37213	26.54858	26.550841	25.90550	25.90559	25.82567	25.82570
2	$1, -3, 2$	$1d_{5/2}, 1d_{3/2}$	29.14606	29.39588	27.39420	27.45807	26.55645	26.57249	25.90582	25.90646	25.82575	25.82591
3	$1, -4, 3$	$1f_{7/2}, 1f_{5/2}$	29.32610	30.08052	27.44078	27.63985	26.56825	26.61874	25.90630	25.90833	25.82587	25.82638
4	$1, -5, 4$	$1g_{9/2}, 1g_{7/2}$	29.56240	31.17924	27.50263	27.94770	26.58397	26.69840	25.90693	25.91155	25.82603	25.82719
1	$2, -2, 1$	$2p_{3/2}, 2p_{2/2}$	31.16455	31.20131	28.43098	28.44013	27.08134	27.08361	26.01178	26.01187	25.87879	25.87881
2	$2, -3, 2$	$2d_{5/2}, 2d_{3/2}$	31.28299	31.53212	28.46176	28.52568	27.08917	27.10523	26.01210	26.01274	25.87887	25.87903
3	$2, -4, 3$	$2f_{7/2}, 2f_{5/2}$	31.45871	32.20983	28.50779	28.70668	27.10090	27.151391	26.01257	26.01460	25.87899	25.87950
4	$2, -5, 4$	$2g_{9/2}, 2g_{7/2}$	31.68950	33.30007	28.56890	29.01332	27.11652	27.230922	26.01320	26.01782	25.87915	25.88030

and $N_{n\kappa}$ is the normalization constant. Further, the lower-spinor component of the wave function can be calculated by using

$$G_{n\kappa}(r) = \frac{1}{M + E_{n\kappa} - C_s}\left(\frac{d}{dr} + \frac{\kappa}{r}\right) F_{n\kappa}(r)\,, \tag{19}$$

where $E \neq -M + C_s$ and in the presence of the exact spin symmetry ($C_s = 0$), only positive energy states do exist.

2.2. *p-spin symmetric limit*

Ginocchio showed that there is p-spin symmetry in case when the relationship between the vector potential and the scalar potential is given by $V(r) = -S(r)$.[7] Further, Meng *et al.* showed that if $\frac{d[V(r)+S(r)]}{dr} = \frac{d\Sigma(r)}{dr} = 0$, then $\Sigma(r) = C_{ps} =$ constant, for which the p-spin symmetry is exact in the Dirac equation.[55–58] Thus, choosing the $\Delta(r)$ as tPT potential, Eq. (6) under this symmetry becomes

$$\left[\frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} - \tilde{\gamma}\left(\frac{V_1}{\sin^2(\alpha r)} + \frac{V_2}{\cos^2(\alpha r)}\right) - \tilde{\beta}^2\right] G_{n\kappa}(r) = 0\,, \tag{20a}$$

$$\tilde{\gamma} = E_{n\kappa} - M - C_{ps} \quad \text{and} \quad \tilde{\beta}^2 = (M + E_{n\kappa})(M - E_{n\kappa} + C_{ps})\,, \tag{20b}$$

where $\kappa = -\tilde{l}$ and $\kappa = \tilde{l} + 1$ for $\kappa < 0$ and $\kappa > 0$, respectively. Employing the new approximation for the spin-orbit pseudo-centrifugal term, $\kappa(\kappa-1)/r^2$, i.e., Eq. (8), the p-spin Dirac equation (20a) can be written as

$$\left[\frac{d^2}{dr^2} - \kappa(\kappa-1)\alpha^2\left(d_0 + \frac{1}{\sin^2(\alpha r)}\right) - \tilde{\gamma}\left(\frac{V_1}{\sin^2(\alpha r)} + \frac{V_2}{\cos^2(\alpha r)}\right) - \tilde{\beta}^2\right] \times G_{n\kappa}(r) = 0\,. \tag{21}$$

To avoid repetition, the negative energy solution of Eq. (21), the p-spin symmetric case can be readily obtained directly via the spin symmetric solution throughout the following parametric mappings:

$$\begin{gathered} F_{n\kappa}(r) \to G_{n\kappa}(r), \quad \kappa \to \kappa - 1, \quad V(r) \to -V(r) \quad (\text{i.e. } V_1 \to -V_1, V_2 \to -V_2)\,, \\ E_{n\kappa} \to -E_{n\kappa}, \quad C_s \to -C_{ps}\,. \end{gathered} \tag{22}$$

Following the previous procedure, one can obtain the p-spin symmetric energy equation as

$$-\left(2n+1+\frac{1}{2}\sqrt{1+\frac{4V_2(E_{n\kappa}-M-C_{ps})}{\alpha^2}}+\frac{1}{2}\sqrt{(2\kappa-1)^2+\frac{4V_1(E_{n\kappa}-M-C_{ps})}{\alpha^2}}\right)^2 = \frac{1}{\alpha^2}(M + E_{n\kappa})(M - E_{n\kappa} + C_{ps}) + \kappa(\kappa-1)d_0\,. \tag{23}$$

Furthermore, the lower-spinor component of the wave functions is found as

$$G_{n\kappa}(r) = \tilde{N}_{n,\kappa}(\sin(\alpha r))^{\frac{1}{2}(1+\tilde{\eta}_\kappa)}(\cos(\alpha r))^{\frac{1}{2}(1+\tilde{\delta})} P_n^{(\frac{1}{2}\tilde{\eta}_\kappa, \frac{1}{2}\tilde{\delta})}(\cos(2\alpha r))\,, \tag{24}$$

Table 4. The bound state energy eigenvalues in unit of fm^{-1} of the p-spin symmetry tPT potential for several values of n and κ with $d_0 = 1/12$.

$\tilde{l}$	n,κ	(l,j)	$\alpha=0.8$ NU	$\alpha=0.8$ AP	$\alpha=0.4$ NU	$\alpha=0.4$ AP	$\alpha=0.2$ NU	$\alpha=0.2$ AP	$\alpha=0.04$ NU	$\alpha=0.04$ AP	$\alpha=0.02$ NU	$\alpha=0.02$ AP
1	1, −1,2	$1s_{1/2}, 0d_{3/2}$	−29.02725	−29.06111	−27.36375	−27.37213	−26.54876	−26.55084	−25.90551	−25.90559	−25.82567	−25.82570
2	1, −2,3	$1p_{3/2}, 0f_{5/2}$	−29.15386	−29.39588	−27.39629	−27.45807	−26.55699	−26.57249	−25.90584	−25.90646	−25.82575	−25.82591
3	1, −3,4	$1d_{5/2}, 0g_{7/2}$	−29.34155	−30.08052	−27.44495	−27.63985	−26.56933	−26.61874	−25.90634	−25.90833	−25.82588	−25.82638
4	1, −4,5	$1f_{7/2}, 0h_{9/2}$	−29.58785	−31.17924	−27.50956	−27.94770	−26.58577	−26.69840	−25.90701	−25.91155	−25.82605	−25.82719
1	2, −1,2	$2s_{1/2}, 1d_{3/2}$	−31.16696	−31.20131	−28.43165	−28.44013	−27.08151	−27.08361	−26.01179	−26.01187	−25.87879	−25.87881
2	2, −2,3	$2p_{3/2}, 1f_{5/2}$	−31.29020	−31.53212	−28.46376	−28.52568	−27.08969	−27.10523	−26.01212	−26.01274	−25.87887	−25.87903
3	2, −3,4	$2d_{5/2}, 1g_{7/2}$	−31.47301	−32.20983	−28.51179	−28.70668	−27.10195	−27.15139	−26.01262	−26.01460	−25.87810	−25.87950
4	2, −4,5	$2f_{7/2}, 1h_{9/2}$	−31.71308	−33.30007	−28.57555	−29.01332	−27.11828	−27.23092	−26.01328	−26.01782	−25.87916	−25.88030

Table 5. The p-spin symmetric bound state energy eigenvalues (fm^{-1}) of the tPT potential for several values of n and κ with $d_0 = 0$.

$\tilde{l}$	n,κ	(l,j)	$\alpha=0.8$ NU	$\alpha=0.8$ AP	$\alpha=0.4$ NU	$\alpha=0.4$ AP	$\alpha=0.2$ NU	$\alpha=0.2$ AP	$\alpha=0.04$ NU	$\alpha=0.04$ AP	$\alpha=0.02$ NU	$\alpha=0.02$ AP
1	1, −1,2	$1s_{1/2}, 0d_{3/2}$	−29.02464	−29.06111	−27.36305	−27.37213	−26.54858	−26.55084	−25.90550	−25.90559	−25.82567	−25.82570
2	1, −2,3	$1p_{3/2}, 0f_{5/2}$	−29.14606	−29.39588	−27.39420	−27.45807	−26.55645	−26.57249	−25.90582	−25.90646	−25.82575	−25.82591
3	1, −3,4	$1d_{5/2}, 0g_{7/2}$	−29.32610	−30.08052	−27.44078	−27.63985	−26.56825	−26.61874	−25.90630	−25.90833	−25.82587	−25.82638
4	1, −4,5	$1f_{7/2}, 0h_{9/2}$	−29.56240	−31.17924	−27.50263	−27.94770	−26.58397	−26.69840	−25.90693	−25.91155	−25.82603	−25.82719
1	2, −1,2	$2s_{1/2}, 1d_{3/2}$	−31.16455	−31.20131	−28.43098	−28.44013	−27.08134	−27.08361	−26.01178	−26.01187	−25.87879	−25.87881
2	2, −2,3	$2p_{3/2}, 1f_{/2}$	−31.28299	−31.53212	−28.46176	−28.52568	−27.08917	−27.10523	−26.01210	−26.01274	−25.87887	−25.87903
3	2, −3,4	$2d_{5/2}, 1g_{7/2}$	−31.45871	−32.20983	−28.50779	−28.70668	−27.10090	−27.15139	−26.01257	−26.01460	−25.87899	−25.87950
4	2, −4,5	$2f_{7/2}, 1h_{9/2}$	−31.68950	−33.30007	−28.56890	−29.01332	−27.11652	−27.23092	−26.01320	−26.01782	−25.87915	−25.88030

with

$$\tilde{\eta}_\kappa = \sqrt{(2\kappa-1)^2 + \frac{4V_1(E_{n\kappa} - M - C_{ps})}{\alpha^2}}, \quad \tilde{\delta} = \sqrt{1 + \frac{4V_2(E_{n\kappa} - M - C_{ps})}{\alpha^2}}, \tag{25}$$

where $\tilde{N}_{n\kappa}$ is the normalization constant.

In Tables 4 and 5, we give the numerical results for the p-spin symmetric case. In this case, we take the set of parameter values, $M = 10$ fm^{-1}, $V_1 = -5.0$ fm^{-1}, $V_2 = -3.0$ fm^{-1}, $c_{ps} = 0$ fm^{-1} and $\alpha = 0.8, 0.4, 0.2, 0.04, 0.02$. We observe the degeneracy in the following doublets $(1s_{1/2}, 0d_{3/2})$, $(1p_{3/2}, 0f_{5/2})$, $(1d_{5/2}, 0g_{7/2})$, $(1f_{7/2}, 0h_{9/2})$, and so on. Thus, each pair is considered as p-spin doublet and has negative energy.[59,64] Furthermore, one can also see the exact numerical energy eigenvalues obtained via AP method to test the accuracy of the present approximate solutions found by using the NU method. As shown in Tables 4 and 5, the accuracy of our results is high when compared with the exact ones in the low screening regime (small values of α). On the other hand, the upper-spinor component of the Dirac wave function can be calculated by

$$F_{n\kappa}(r) = \frac{1}{M - E_{n\kappa} + C_{ps}} \left(\frac{d}{dr} - \frac{\kappa}{r}\right) G_{n\kappa}(r), \tag{26}$$

where $E \neq M + C_{ps}$ and in the presence of the exact p-spin symmetry ($C_{ps} = 0$), only negative energy states do exist.

2.3. *The nonrelativistic limiting case*

In this section, we study the energy eigenvalue equation (12) and upper-spinor component of wave function (17) of the Dirac-tPT problem under the nonrelativistic limits: $C_s = 0, \kappa \to l$, $E_{n\kappa} - M \simeq E_{nl}$ and $M + E_{n\kappa} \simeq 2m$. Thereby, we obtain the energy equation of the Schrödinger equation with any arbitrary orbital state for the tPT potential as

$$E_{nl} = \frac{\hbar^2\alpha^2 l(l+1)d_0}{2m} + \frac{2\hbar^2\alpha^2}{m}\left[n + \frac{1}{2} + \frac{1}{4}\left(\sqrt{(2l+1)^2 + \frac{8mV_1}{\hbar^2\alpha^2}} + \sqrt{1 + \frac{8mV_2}{\hbar^2\alpha^2}}\right)\right]^2. \tag{27}$$

In the limit when $\alpha \to 0$, the vibration-rotation energy formula (27) reduces into a constant value:

$$\lim_{\alpha\to 0} E_{nl} = (\sqrt{V_1} + \sqrt{V_2})^2. \tag{28}$$

Further, there is no loss of generality if $d_0 = 0$, then Eq. (27) becomes

$$E_{nl} = \frac{2\hbar^2\alpha^2}{m}\left[n + \frac{1}{2} + \frac{1}{4}\left(\sqrt{(2l+1)^2 + \frac{8mV_1}{\hbar^2\alpha^2}} + \sqrt{1 + \frac{8mV_2}{\hbar^2\alpha^2}}\right)\right]^2, \tag{29}$$

Table 6. The bound state energy levels E_{nl} of the Schrödinger equation for the tPT potential.

State (n,l)	E_{nl} $M = 10.0$ fm^{-1}, $V_1 = 5$ fm^{-1}, $V_2 = 3$ fm^{-1} (Ref. 15) and $d_0 = \frac{1}{12}$					
	$\alpha = 1.2$	$\alpha = 0.8$	$\alpha = 0.4$	$\alpha = 0.2$	$\alpha = 0.02$	$\alpha = 0.002$
1s (Ref. 21)	22.87051710	20.32991862	17.95616357	16.83082621	15.85264289	15.75661628
2s (Ref. 21)	28.29143398	23.68420415	19.50420742	17.57271070	15.92394680	15.76371786
2p	28.64395419	23.82847894	19.53712286	17.58054181	15.92402153	15.76371860
3s (Ref. 21)	34.28835086	27.2944896	21.11625126	18.33059518	15.99541071	15.77082105
3p	34.67512504	27.44896381	21.15044543	18.33858626	15.99548560	15.77082179
3d	35.43921159	27.75631556	21.21875330	18.35456399	15.99563534	15.77082328
4s (Ref. 21)	40.86126774	31.16077522	22.79229510	19.10447967	16.06703463	15.77792584
4p	41.28229584	31.32544868	22.82776800	19.11263070	16.06710967	15.77792658
4d	42.11348590	31.65300783	22.89862721	19.12892817	16.06725974	15.77792806
4f	43.33519178	32.14003977	23.00470171	19.15336297	16.06748485	15.77793030

Table 7. The bound state energy levels E_{nl} of the Schrödinger equation for the tPT potential.

State (n,l)	E_{nl} $M = 10$ fm^{-1}, $V_1 = 5$ fm^{-1}, $V_2 = 3$ fm^{-1} (Ref. 14) and $d_0 = 0$					
	$\alpha = 1.2$	$\alpha = 0.8$	$\alpha = 0.4$	$\alpha = 0.2$	$\alpha = 0.02$	$\alpha = 0.002$
1s (Ref. 21)	22.87051710	20.32991862	17.95616357	16.83082621	15.85264289	15.75661628
2s (Ref. 21)	28.29143398	23.68420415	19.50420742	17.57271070	15.92394680	15.76371786
2p	28.60696804	23.81346538	19.53372130	17.57973494	15.92401384	15.76371853
3s (Ref. 21)	34.28835086	27.2944896	21.11625126	18.33059518	15.99541071	15.77082105
3p	34.63442822	27.43284957	21.14690619	18.33776218	15.99547790	15.77082171
3d	35.31564933	27.70768323	21.20811765	18.35209065	15.99561226	15.77082304
4s (Ref. 21)	40.86126774	31.16077522	22.79229510	19.10447967	16.06703463	15.77792584
4p	41.23788840	31.30823378	22.82409108	19.11178941	16.06710195	15.77792650
4d	41.97876941	31.60107057	22.88757844	19.12640318	16.06723660	15.77792783
4f	43.06137688	32.03529723	22.98255025	19.14830965	16.06743857	15.77792984

where $n = 0, 1, 2, \ldots$ and $l = 0, 1, 2, \ldots$ are the vibration and rotation quantum numbers, respectively. To obtain a numerical energy eigenvalues for the present potential model, we take the following set of parameter values; namely, $M = 10\ \text{fm}^{-1}$, $V_1 = 5.0\ \text{fm}^{-1}$, $V_2 = 3.0\ \text{fm}^{-1}$ and $\alpha = 1.2, 0.8, 0.4, 0.2, 0.02, 0.002$.[19,21,65] As seen from Tables 5 and 6, in the limit when potential range parameter α approaches zero, the energy eigenvalues approaches a constant value given by Eq. (28). In Tables 6 and 7, we take $d_0 = 1/12$ and $d_0 = 0$, respectively.

Also we can get the radial wave functions of the Schrödinger equation with tPT potential as

$$R_{n,l}(s) = s^{\frac{1}{4}\left[1+\sqrt{(2l+1)^2+\frac{8mV_1}{\hbar^2\alpha^2}}\,\right]}(1-s)^{\frac{1}{4}\left(1+\sqrt{1+\frac{8mV_2}{\hbar^2\alpha^2}}\,\right)} \times P_n^{\left(\frac{1}{2}\sqrt{(2l+1)^2+\frac{8mV_1}{\hbar^2\alpha^2}},\frac{1}{2}\sqrt{1+\frac{8mV_2}{\hbar^2\alpha^2}}\,\right)}(1-2s)\,. \tag{30}$$

Inserting $s = \sin^2(\alpha r)$ in the above equation, we can obtain

$$R_{n,l}(r) = N_{nl}(\sin(\alpha r))^{(1+\eta_l)/2}(\cos(\alpha r))^{(1+\delta)/2}P_n^{(\eta_l/2,\delta/2)}(\cos(2\alpha r))\,, \tag{31a}$$

$$\eta_l = \sqrt{(2l+1)^2+\frac{8mV_1}{\hbar^2\alpha^2}}, \quad \delta = \sqrt{1+\frac{8mV_2}{\hbar^2\alpha^2}} \tag{31b}$$

where N_{nl} is a normalization factor to be calculated from the normalization conditions.

To conclude, we need to mention that one of the applications of the nonrelativistic tPT model is in molecular physics, to calculate the energy spectrum of the diatomic molecules.[65]

3. Concluding Remarks

In this work, we have investigated the bound state solutions of the Dirac equation with trigonometric Pöschl–Teller potential for any spin-orbit quantum number κ. By making an appropriate approximation to deal with the spin-orbit centrifugal (pseudo-centrifugal) coupling term, we have obtained the approximate energy eigenvalue equation and the unnormalized two components of the radial wave functions expressed in terms of the Jacobi polynomials using the NU method. Also, we obtained the exact numerical energy eigenvalues via AP method to test the accuracy of the present solutions. Furthermore, we have obtained the nonrelativistic solutions for the rotation-vibration energy eigenvalues and corresponding radial wave function of the spin-0 particle moving under the influence of the tPT field.

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Appendix A. Parametric Generalization of the NU Method

The NU method is used to solve second-order differential equations with an appropriate coordinate transformation $s = s(r)$ (Ref. 24)

$$\psi_n''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi_n'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi_n(s) = 0\,, \tag{A.1}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most of second degree, and $\tilde{\tau}(s)$ is a first-degree polynomial. To make the application of the NU method simpler and direct without need to check the validity of solution. We present a shortcut for the method. So, at first we write the general form of the Schrödinger-like equation (A.1) in a more general form applicable to any potential as follows[27]:

$$\psi_n''(s) + \left(\frac{c_1 - c_2 s}{s(1 - c_3 s)}\right)\psi_n'(s) + \left(\frac{-\xi_1 s^2 + \xi_2 s - \xi_3}{s^2(1 - c_3 s)^2}\right)\psi_n(s) = 0\,, \tag{A.2}$$

satisfying the wave functions

$$\psi_n(s) = \varphi(s) y_n(s)\,. \tag{A.3}$$

Comparing (A.2) with its counterpart (A.1), we obtain the following identifications:

$$\tilde{\tau}(s) = c_1 - c_2 s, \quad \sigma(s) = s(1 - c_3 s), \quad \tilde{\sigma}(s) = -\xi_1 s^2 + \xi_2 s - \xi_3\,. \tag{A.4}$$

Following the NU method,[24] we obtain the following[27]:

(i) the relevant constant:

$$\begin{aligned}
&c_4 = \frac{1}{2}(1 - c_1), && c_5 = \frac{1}{2}(c_2 - 2c_3),\\
&c_6 = c_5^2 + A, && c_7 = 2c_4 c_5 - B,\\
&c_8 = c_4^2 + C, && c_9 = c_3(c_7 + c_3 c_8) + c_6,\\
&c_{10} = c_1 + 2c_4 + 2\sqrt{c_8} - 1 > -1, && c_{11} = 1 - c_1 - 2c_4 + \frac{2}{c_3}\sqrt{c_9} > -1, \quad c_3 \neq 0,\\
&c_{12} = c_4 + \sqrt{c_8} > 0, && c_{13} = -c_4 + \frac{1}{c_3}(\sqrt{c_9} - c_5) > 0, \quad c_3 \neq 0.
\end{aligned} \tag{A.5}$$

(ii) The essential polynomial functions:

$$\pi(s) = c_4 + c_5 s - [(\sqrt{c_9} + c_3\sqrt{c_8})s - \sqrt{c_8}]\,, \tag{A.6}$$

$$k = -(c_7 + 2c_3 c_8) - 2\sqrt{c_8 c_9}\,, \tag{A.7}$$

$$\tau(s) = c_1 + 2c_4 - (c_2 - 2c_5)s - 2[(\sqrt{c_9} + c_3\sqrt{c_8})s - \sqrt{c_8}]\,, \tag{A.8}$$

$$\tau'(s) = -2c_3 - 2(\sqrt{c_9} + c_3\sqrt{c_8}) < 0\,. \tag{A.9}$$

(iii) The energy equation:

$$\begin{aligned}
&(c_2 - c_3)n + c_3 n^2 - (2n+1)c_5 + (2n+1)(\sqrt{c_9} + c_3\sqrt{c_8})\\
&\quad + c_7 + 2c_3 c_8 + 2\sqrt{c_8 c_9} = 0\,.
\end{aligned} \tag{A.10}$$

(iv) The wave functions

$$\rho(s) = s^{c_{10}}(1 - c_3 s)^{c_{11}}\,, \tag{A.11}$$

$$\phi(s) = s^{c_{12}}(1 - c_3 s)^{c_{13}}, \quad c_{12} > 0, \quad c_{13} > 0\,, \tag{A.12}$$

$$y_n(s) = P_n^{(c_{10},c_{11})}(1 - 2c_3 s), \quad c_{10} > -1, \quad c_{11} > -1\,, \tag{A.13}$$

$$\psi_{n\kappa}(s) = N_{n\kappa} s^{c_{12}}(1 - c_3 s)^{c_{13}} P_n^{(c_{10},c_{11})}(1 - 2c_3 s)\,, \tag{A.14}$$

where $P_n^{(\mu,\nu)}(x), \mu > -1, \nu > -1$ and $x \in [-1, 1]$ are Jacobi polynomials with

$$P_n^{(\alpha,\beta)}(1 - 2s) = \frac{(\alpha + 1)_n}{n!} {}_2F_1(-n,\, 1 + \alpha + \beta + n;\, \alpha + 1;\, s) \tag{A.15}$$

and $N_{n\kappa}$ is a normalization constant. Also, the above wave functions can be expressed in terms of the hypergeometric function as

$$\psi_{n\kappa}(s) = N_{n\kappa} s^{c_{12}}(1 - c_3 s)^{c_{13}} {}_2F_1(-n, 1 + c_{10} + c_{11} + n; c_{10} + 1; c_3 s)\,, \tag{A.16}$$

where $c_{12} > 0, c_{13} > 0$ and $s \in [0, 1/c_3]$, $c_3 \neq 0$.

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