

On the fourth order finite screw system: general displacement of a point in a rigid body

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Summary. In this paper, the displacement of a point on a rigid body which moves from one spatial position to another is considered by the concept of screw matrix. Since there are ∞^3 possible screws to achieve such displacement, using Parkin's definition (measure) for pitch, one can represent this infinite set of screws as a 4-system of screws. This has been demonstrated by Huang and Roth by the use of screw triangle [1]. In this paper, however, without any assumption for the expression of pitch we ask the question that whether there exist different appropriate measure of pitch other than Parkin's such that all the available finite screws also form a 4-system. In fact by the use of screw matrix concept, a general expression of the pitch is derived so that the screws realize the stated displacement form a 4-system. It is shown that Parkin's pitch, plus an arbitrary constant, is the only appropriate measure of pitch. To arrive at this conclusion, a special coordinate system is selected. This selection of the coordinate system also makes it possible to yield several new geometrical properties of the set of ∞^3 screws.

1 Introduction

The displacement of a rigid body in space can be described uniquely as the combination of a rotation (angle) about and a translation along a unique screw axis. In this displacement the ratio of the translation to the rotation is called the cardinal pitch of the finite screw [2]. The screw displacement is usually referred to as finite twist displacement [1]. The mathematical counterpart of a physical screw which would produce a specified spatial displacement of a rigid body is the screw matrix. Beggs [3] has derived an expression for the screw matrix by the use of two different coordinate systems. However the expression given appears to be rather a complicated 4×4 matrix which does not exhibit any immediate clues about the physical properties of the screw. Ramahi and Tokad [4] have introduced a simple and compact expression for a screw matrix expressed in a single fixed coordinate system.

As it is known, the finite displacement of a rigid body can be represented completely by six independent parameters. When less parameters are specified, the displacement of the rigid body is called incompletely specified displacement [1]. The finite displacements of a line with a point, a line, and a point in a rigid body are all incompletely specified displacements. Indeed, only five, four and three independent parameters are sufficient to specify, respectively, the displacements of a line with a point, a line, and a point in a rigid body. Transformation matrices associated with the representation of these displacements involve, respectively, one, two and three free parameters. Hence, there are ∞^1 , ∞^2 and ∞^3 available screws to perform these displacements. Previously, Tsai and Roth [5] have studied all possible screws associated with the incompletely specified displacements of a line with a point, a line only, a point, and a point on a plane. They have investigated these problems based on the concept of the screw triangle

[2]–[6]. Sticher [7], by choosing a special coordinate system, has derived the finite screw cylindroid associated with the displacement of a directed line with a point on it. He followed a geometric approach different than that in [5]. Until recently, it was believed that finite twists did not have linear properties. Later Parkin [8] has defined a new pitch for a finite screw as the ratio of one-half the translation to the tangent of one-half the rotation, and has shown that under his new definition of pitch the finite screw cylindroid can be represented by the linear combinations of two bases screws. In other words, Parkin's works shows, under his definition of pitch, that all available finite screws for displacing a line with a point form a vector space, namely, a 2-system of screws. Although Parkin states the expression of pitch without any reasoning, he shows that this new definition of pitch constitutes a sufficient condition for the existence of a 2-system which is embedded in the finite screw cylindroid. Huang and Roth [1] by using Parkin's definition of pitch, studied the screw systems for the incompletely specified displacement problems tackled originally by Tsai and Roth [5]. In particular, they have shown that with the use of Parkin's pitch, any of the ∞^2 possible screws for the displacement of a line in a rigid body cannot be expressed as a linear combination of three bases screws. On the other hand, they derived an explicit analytical expressions for a 2-system and 4-system of screws corresponding, respectively, to the displacements of a line with a point, and a point of a rigid body. Huang and Chen [9], by using Parkin's definition of pitch again, derived the linear representation of the screw triangle. Using this linear representation for the screw triangle, they also performed the finite kinematics analysis of multi-link serial chains to demonstrate a unification of finite and infinitesimal kinematics. Ramahi and Tokad [4], by introducing a simplified version for a screw matrix and using Parkin's definition of pitch, have obtained a simplified analytical expression for the 2-system of screws resides on the finite screw cylindroid. Ramahi and Tokad [10] also have shown that Parkin's pitch, plus an arbitrary constant which may be taken as zero, is the only appropriate measure (expression) of pitch under which the finite screw cylindroid forms a 2-system. Hence, Parkin's pitch is sufficient, however it may be considered as a special case of the general form given in [10]. Finally, Ramahi and Tokad [11] have extended their approach to derive an expression for the pitch, which is both the necessary and sufficient condition, under which the set of ∞^2 screws associated with the displacement of a line in a moving rigid body form a 3-system.

In this paper, the incompletely specified displacement of a point in a moving rigid body is studied based on linear algebra. Analytic descriptions of all possible screws associated with the general spatial displacement of a point in a moving rigid body are obtained. Without loss of generality, a special coordinate system is selected. With this selection, it is shown that the pitch previously obtained in [10] is the only possible measure of pitch under which the set of ∞^3 screws form a 4-system. This selection of a special coordinate system makes it also possible to display several new geometric properties of the set of ∞^3 screws.

2 Preliminary considerations

A spatial vector \underline{a} , in a fixed Cartesian coordinate system Σ_o , is represented by a 3×1 column matrix \mathbf{a} or equivalently by a 3×3 skew symmetric matrix \mathbf{A} . Therefore, the scalar product $l = \underline{a} \cdot \underline{b}$ and the vector product $\underline{c} = \underline{a} \times \underline{b}$ of two vectors \underline{a} and \underline{b} are denoted in Σ_o by the respective matrix products $l = \mathbf{a}^T \mathbf{b}$ and $\mathbf{c} = \mathbf{A} \mathbf{b}$. The use of such representation usually provides some simplifications in the establishment of vectorial relations [12], [13].

The Plucker representation of a line L in space may be given by a 6×1 column matrix $L = [\mathbf{n} \ \mathbf{m}]^T$, where \mathbf{n} and \mathbf{m} , respectively represent the direction cosine vector of the line with $\mathbf{n}^T \mathbf{n} = 1$, and its moment vector with respect to the origin of Σ_o :

$$N\zeta = -\mathbf{m}. \quad (1)$$

Where ζ represents the position vector of any point on the line. Since $N\mathbf{n} = 0$ and $\mathbf{n}^T N = 0$, then we have $\mathbf{n}^T \mathbf{m} = 0$ which constitutes the constraint on the pair (\mathbf{n}, \mathbf{m}) . When the pair of column matrices \mathbf{n} and \mathbf{m} are given, satisfying the constraint $\mathbf{n}^T \mathbf{m} = 0$, then from Eq. (1) the position vector can be found as

$$\zeta = \lambda \mathbf{n} + N\mathbf{m}. \quad (2)$$

Where λ is an arbitrary real parameter, and the second term represents the perpendicular component, ζ^\perp , of ζ with respect to \mathbf{n} :

$$\zeta^\perp = N\mathbf{m}. \quad (3)$$

A unit screw \wp is a line with unit direction \mathbf{n} and at a radius vector ζ^\perp from the origin together with an associated scalar called the pitch, p , which relates the translation along and the rotation about the screw axis. Therefore, a line alone is a screw with zero pitch. A general (roto-translation) screw can be represented in screw coordinates as

$$\wp = [\mathbf{n}^o \ \mathbf{m}^o]^T = [\mathbf{n} \ \mathbf{m} + p\mathbf{n}]^T \quad (4)$$

hence, with this notation we have

$$\zeta^\perp = N^o \mathbf{m}^o = N\mathbf{m} \quad \text{and} \quad p = \mathbf{n}^T \mathbf{m}^o. \quad (5)$$

Note that in general when $p = 0$, we have a pure rotation screw $\wp = [\mathbf{n} \ \mathbf{m}]^T$ and when $p = \infty$, we have a pure translation screw $\wp = [0 \ \mathbf{n}]^T$.

When a Cartesian coordinate system Σ is attached rigidly to a body and the body is rotating about the origin of the fixed coordinate system Σ_o , the displacement of a point on the body may be determined by the coordinate transformation (rotation) matrix \mathbf{T} which may be expressed by the Rodrigues formula [14], [15]

$$\mathbf{T} = \cos \theta \mathbf{I} + (1 - \cos \theta) \mathbf{m}\mathbf{m}^T + \sin \theta N = \mathbf{I} + \sin \theta N + (1 - \cos \theta) N^2. \quad (6)$$

In this formula, according to Euler's theorem [16], \mathbf{n} is the direction cosine vector of the rotation axis while θ is the rotation angle about this axis and \mathbf{I} represents the unit (identity) matrix. \mathbf{T} is an orthonormal matrix. One can see easily that $(\mathbf{I} - \mathbf{T})\mathbf{n} = 0$.

On the other hand, if Σ is not only rotating but also translating by an amount ξ with respect to Σ_o , then the homogeneous transformation matrix [17], [18] of order four is required to express the general displacement of a rigid body

$$\mathbf{H} = \begin{bmatrix} \mathbf{T} & \xi \\ 0 & 1 \end{bmatrix}. \quad (7)$$

However, by Chasles theorem [16], the same displacement can be obtained by a finite twist (combination of rotation and translation) along a certain screw axis. The mathematical representation of this equivalent finite twist is the following 4×4 screw matrix \mathcal{S} [4], [10], [11]

$$\mathcal{S} = \begin{bmatrix} \mathbf{T}^* & (\mathbf{I} - \mathbf{T}^*)\zeta + \lambda \mathbf{n} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{T} & \xi \\ 0 & 1 \end{bmatrix} \quad (8)$$

where ζ and λ represent, respectively, the position vector of a point on the screw axis and the amount of translation along the same axis.

Given the pair (\mathbf{T}, ξ) , the physical properties $(\theta, \mathbf{n}, \lambda, \zeta^\perp)$ of the screw can be obtained as [4], [10], [11]

$$\begin{aligned}\cos \theta &= \frac{1}{2}(1 - \text{tr } \mathbf{T}); \\ \mathbf{N} &= \frac{1}{2 \sin \theta}(\mathbf{T} - \mathbf{T}^T), \\ \lambda &= \mathbf{n}^T \xi; \\ \zeta^\perp &= \frac{1}{2} \left[\frac{1}{\tan(\theta/2)} \mathbf{I} - \mathbf{N} \right] \mathbf{N} \xi\end{aligned}\tag{9}$$

where $\text{tr}(\cdot)$ indicates the trace of a square matrix (\cdot) , and ζ^\perp is the vector perpendicular from the origin of Σ_o to the screw axis.

Note that successive finite twist displacements can be represented as the product of screw matrices [3]. For instance, the product $\mathbb{S}_3 = \mathbb{S}_2 \mathbb{S}_1$ suggests that there is a single screw \wp_3 , equivalent to two corresponding screws \wp_1 and \wp_2 which are operated on Σ coordinate frame in that order. Therefore, one may determine the physical properties of \wp_3 in terms of those of \wp_1 and \wp_2 .

3 The incompletely specified finite displacement of a point in a moving rigid body

In this section we first discuss the general displacement of a point in a moving rigid body, then for the sake of further simplification of the expressions developed, a special coordinate system will be employed. As shown in Fig. 1, let v_1 and v_2 be the initial and final positions of a point in a moving rigid body. If \mathbf{v}_1 and \mathbf{v}_2 represent, respectively, the position vectors of v_1 and v_2 , then the displacement of the moving rigid body which carries v_1 into v_2 is not unique. In fact there are ∞^3 set of screws that can realize this displacement. We are interested in

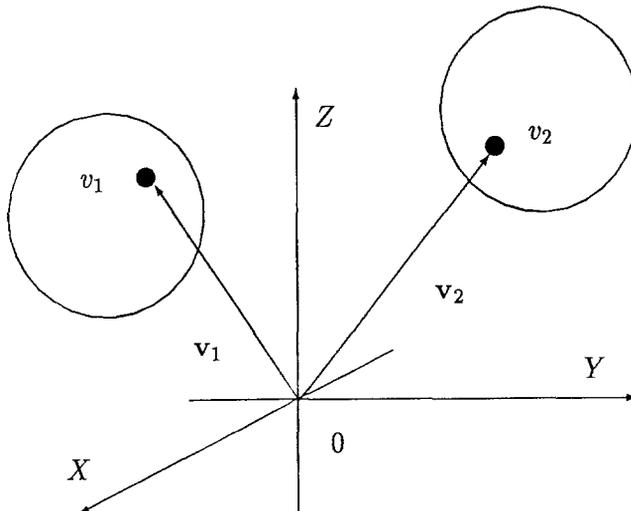


Fig. 1. A general displacement of a point in an arbitrarily displaced rigid body

describing this set of screws. One can consider the displacement in two steps: first, translate the rigid body so that \mathbf{v}_1 coincides with \mathbf{v}_2 , second rotate the rigid body only about \mathbf{v}_2 by a zero translation screw whose direction and angle of rotation are free parameters. Using Eq. (7), the displacement in the first step can be represented by homogeneous matrix \mathbf{H}_{12} with submatrices $\mathbf{T}_{12} = \mathbf{I}$ and $\xi_{12} = (\mathbf{v}_2 - \mathbf{v}_1)$:

$$\mathbf{H}_{12} = \begin{bmatrix} \mathbf{I} & (\mathbf{v}_2 - \mathbf{v}_1) \\ 0 & 1 \end{bmatrix}.$$

In this special case \mathbf{H}_{12} actually is the screw matrix corresponding to this transformation. The displacement in the second step can be represented by the screw $\wp_{2(n,\alpha)}$ whose physical properties are: $\mathbf{n}_{2(n,\alpha)} = [x \ y \ z]^T$ (the direction cosine vector of its axis), $\theta_{2(n,\alpha)} = \alpha$ (the angle of rotation about this axis), $\lambda = 0$ (the amount of translation along this axis) and \mathbf{v}_2 (its location). Hence from Eq. (8) the screw matrix corresponding to $\wp_{2(n,\alpha)}$ is given by

$$\mathbb{S}_{2(n,\alpha)} = \begin{bmatrix} \mathbf{T}_{2(n,\alpha)} & (\mathbf{I} - \mathbf{T}_{2(n,\alpha)}) \mathbf{v}_2 \\ 0 & 1 \end{bmatrix}$$

where

$$\mathbf{T}_{2(n,\alpha)} = \mathbf{I} + \sin \alpha \mathbf{N}_{2(n,\alpha)} + (1 - \cos \alpha) \mathbf{N}_{2(n,\alpha)}^2.$$

However, the entire displacement of the rigid body can be represented by a screw $\wp_{1(n,\alpha)}$, with the corresponding screw matrix

$$\begin{aligned} \mathbb{S}_{1(n,\alpha)} &= \mathbb{S}_{2(n,\alpha)} \mathbf{H}_{12} = \begin{bmatrix} \mathbf{T}_{2(n,\alpha)} & (\mathbf{I} - \mathbf{T}_{2(n,\alpha)}) \mathbf{v}_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & (\mathbf{v}_2 - \mathbf{v}_1) \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{T}_{1(n,\alpha)} & (\mathbf{I} - \mathbf{T}_{1(n,\alpha)}) \zeta_{1(n,\alpha)} + \lambda_{1(n,\alpha)} \mathbf{n}_{1(n,\alpha)} \\ 0 & 1 \end{bmatrix} \end{aligned} \quad (15)$$

From the equality of the matrices we have the relations

$$\mathbf{T}_{1(n,\alpha)} = \mathbf{T}_{2(n,\alpha)}$$

and

$$(\mathbf{I} - \mathbf{T}_{1(n,\alpha)}) \zeta_{1(n,\alpha)} + \lambda_{1(n,\alpha)} \mathbf{n}_{1(n,\alpha)} = \mathbf{v}_2 - \mathbf{T}_{2(n,\alpha)} \mathbf{v}_1 = \xi_{1(n,\alpha)}.$$

Now, from the last two equations and by making use of Eq. (8), the physical properties, $(\theta_{1(n,\alpha)}, \mathbf{n}_{1(n,\alpha)}, \lambda_{1(n,\alpha)}, \zeta_{1(n,\alpha)}^\perp)$, of the non-unique screw carrying \mathbf{v}_1 in \mathbf{v}_2 , will be of the form

$$\theta_{1(n,\alpha)} = \theta_{2(n,\alpha)} = \alpha;$$

$$\mathbf{n}_{1(n,\alpha)} = \mathbf{n}_{2(n,\alpha)} = [x \ y \ z]^T;$$

$$\lambda_{1(n,\alpha)} = \mathbf{n}_{2(n,\alpha)}^T (\mathbf{v}_2 - \mathbf{T}_{2(n,\alpha)} \mathbf{v}_1) = \mathbf{n}_{2(n,\alpha)}^T (\mathbf{v}_2 - \mathbf{v}_1);$$

$$\zeta_{1(n,\alpha)}^\perp = \frac{1}{2 \tan(\alpha/2)} \mathbf{N}_{2(n,\alpha)} [(\mathbf{v}_2 - \mathbf{v}_1) - \mathbf{N}_{2(n,\alpha)} (\mathbf{v}_2 + \mathbf{v}_1)].$$

(16)

Note that $x^2 + y^2 + z^2 = 1$, and Eq. (16) gives the analytic description of ∞^3 set of screws each realizes the same general displacement of a point in a moving rigid body.

At this stage, for the sake of simplification of the expressions in Eq. (16), a special coordinate system shown in Fig. 2 is selected. Based on this coordinate system we have $\mathbf{v}_1 = [d \ 0 \ 0]^T$ and $\mathbf{v}_2 = [-d \ 0 \ 0]^T$. Hence, expressions in Eq. (16) simplify to

$$\begin{aligned}\theta_{1(n,\alpha)} &= \alpha; \\ \mathbf{n}_{1(n,\alpha)} &= [x \ y \ z]^T; \\ \lambda_{1(n,\alpha)} &= -2dx; \\ \zeta_{1(n,\alpha)}^\perp &= \frac{d}{\tan(\alpha/2)} [0 \ -z \ y]^T.\end{aligned}\tag{17}$$

The minus sign in the expression of $\lambda_{1(n,\alpha)}$ indicates the translation of the rigid body is in the opposite direction of that of the screw axis.

4 Some notable geometrical properties of the ∞^3 set of screws

From Eq. (17) one can deduce easily that:

(a) The vector $\zeta_{1(n,\alpha)}^\perp$ is perpendicular to the screw axis and lies on the OYZ plane. Therefore, the projection, $\mathbf{n}_{1(n,\alpha)}^{YZ}$, of $\mathbf{n}_{1(n,\alpha)}$ on OYZ plane is also perpendicular to $\zeta_{1(n,\alpha)}^\perp$. Note also that a point B with the position vector $\mathbf{n}_{1(n,\alpha)}^{YZ}$ lies in the interior of the unit circle C in the OYZ plane.

(b) Let the direction vector, $\mathbf{n}_{1(n,\alpha)} = [x \ y \ z]^T$, and the rotation angle, $(\theta_{1(n,\alpha)} = \alpha)$, be chosen arbitrarily, then the location, $\zeta_{1(n,\alpha)}^\perp$, and the translation displacement, $\lambda_{1(n,\alpha)}$, are determined uniquely from Eq. (17). In this case, with $x \neq 0$, the screw axis cuts OYZ plane at a point Q which is exactly the tip of the vector $\zeta_{1(n,\alpha)}^\perp$.

(c) If we consider a point B inside the unit circle C , whose position vector corresponds to the projection of $\mathbf{n}_{1(n,\alpha)}$ on OYZ plane, then the location of Q is determined by the angle α ($-\pi < \alpha < \pi$). That is, in this case, the locus of the point Q is a straight line Δ passing through the origin. Note also that, because of the relation $x = \mp\sqrt{1 - (y^2 + z^2)}$, there will be two screw axes passing through the point Q , these two screw axes are the mirror images of each other with respect to the OYZ plane. Moreover, if the point B is selected such that $z^2 - y^2 = 1/2$, then the two mirror image screw axes become perpendicular to each others along the line Δ .

(d) If α is selected, and if $z^2 + y^2 = a^2$, where $0 < a^2 \leq 1$, then the projection of the screw axes in the OYZ plane envelopes a circle F of radius $r = ad/\tan(\alpha/2)$ with its center at the origin. If the screw is a pure rotational screw, then from Eq. (17) we have $x = 0$ (i.e., $a = 1$) and hence the radius of F becomes $r = d/\tan(\alpha/2)$. Moreover, if $\alpha = \pm\pi$, then the radius of F becomes zero.

(e) The axes of all pure rotational screws lie in the OYZ plane.

(f) The axes of all pure translational screws are perpendicular to the OYZ plane. On the other hand, the position of pure translational screw axes is defined by any point on the OYZ plane.

5 The representation of the set of ∞^3 screws as a 4-system

In this section we search for a suitable expression of pitch under which the set of ∞^3 screws considered above form a 4-system. From Eq. (4) any screw $\wp_{1(n,\alpha)}$ in this set can be expressed as

$$\wp_{1(n,\alpha)} = [\mathbf{n}_{1(n,\alpha)} - N_{1(n,\alpha)}\zeta_{1(n,\alpha)}^\perp + P_{1(n,\alpha)}\mathbf{n}_{1(n,\alpha)}]^T. \quad (18)$$

Substitution of the expressions of $\mathbf{n}_{1(n,\alpha)}$ and $\zeta_{1(n,\alpha)}^\perp$ from Eq. (17) into Eq. (18) gives

$$\wp_{1(n,\alpha)} = \left[\begin{array}{c} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ \left(\frac{xd}{\tan(\alpha/2)} + P_{1(n,\alpha)} \right) \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \frac{d}{\tan(\alpha/2)} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \end{array} \right]. \quad (19)$$

Now, in order that the screws in Eq. (18) to form a 4-system, they must be put into the following form (see also [1])

$$\wp_{1(n,\alpha)} = f_a(n,\alpha)\wp_a + f_b(n,\alpha)\wp_b + f_c(n,\alpha)\wp_c + f_d(n,\alpha)\wp_d \quad (20)$$

where $f_a(n,\alpha)$, $f_b(n,\alpha)$, $f_c(n,\alpha)$ and $f_d(n,\alpha)$ are scalar functions of the free parameters α and \mathbf{n} , while \wp_a , \wp_b , \wp_c and \wp_d are constant screws. One can easily see that for the expression in Eq. (18) to be put into the form in Eq. (20), its explicit form given in Eq. (19) must necessary be expressed as

$$\wp_{1(n,\alpha)} = \left[\begin{array}{c} x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{d}{\tan(\alpha/2)} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ x \begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ R \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ R \end{bmatrix} - \frac{d}{\tan(\alpha/2)} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \end{array} \right] \quad (21)$$

where

$$R = \frac{xd}{\tan(\alpha/2)} + P_{1(n,\alpha)} \quad (22)$$

which must be an arbitrary real constant, hence we have

$$P_{1(n,\alpha)} = \frac{-xd}{\tan(\alpha/2)} + R. \quad (23)$$

Considering the expression $\lambda_{1(n,\alpha)}$ given in Eq. (17), we finally obtain

$$P_{1(n,\alpha)} = \frac{\lambda_{1(n,\alpha)}}{2 \tan(\theta_{1(n,\alpha)}/2)} + R. \quad (24)$$

The first term of the right hand side of Eq. (24) is exactly the definition of pitch introduced by Parkin. Therefore, Parkin's pitch, plus an arbitrary real constant R , is the only pitch under which the ∞^3 set of screws associated with the finite displacement of a point in an arbitrarily displaced rigid body forms a 4-system. One can see from Eq. (5) and Eq. (17) that the physical properties $(\theta_{1(n,\alpha)}, \mathbf{n}_{1(n,\alpha)}, \lambda_{1(n,\alpha)}, \zeta_{1(n,\alpha)}^\perp)$ of the screw $\wp_{1(n,\alpha)}$ are not affected by the arbitrary

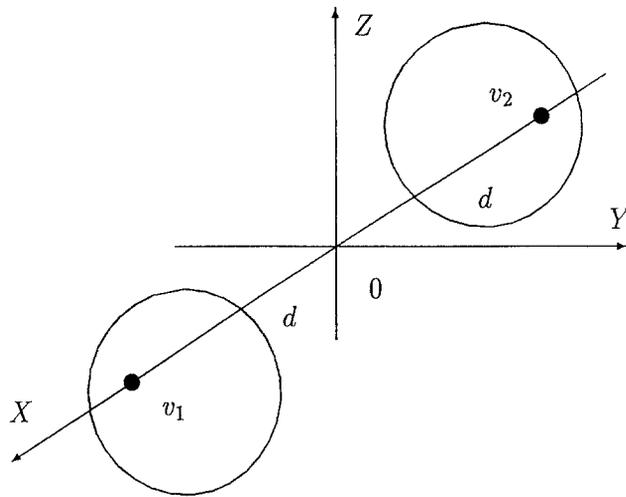


Fig. 2. A coordinate system in which the initial and final positions of the point on the OX axis

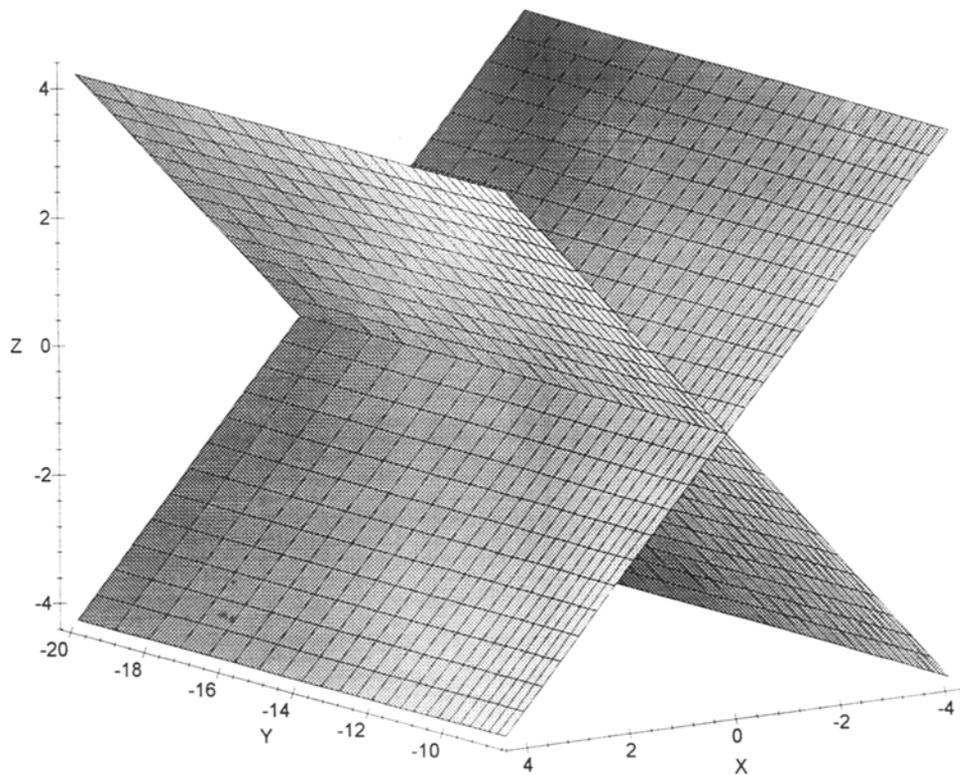


Fig. 3. A plot of W with $d = 5$, $y = 0$, $z = 1/\sqrt{2}$ and $\pi/9 \leq \alpha \leq \pi/4$

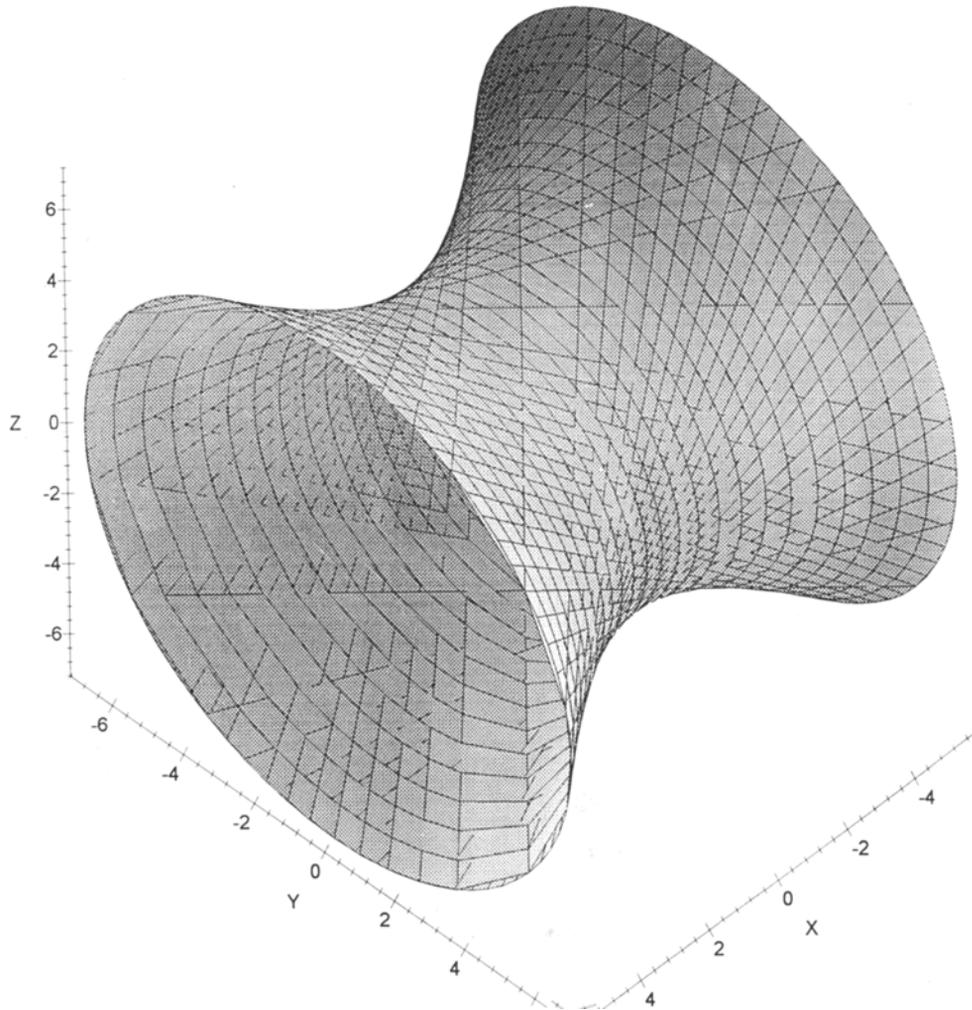


Fig. 4. Plot of W with $d = 4\sqrt{2}$, $y^2 + z^2 = 1/2$, where $-1/\sqrt{2} \leq 1/\sqrt{2}$, and $\alpha = \pi/2$

constant appearing in the expression of the pitch, hence the constant R can always be taken as zero so that the pitch in Eq. (24) reduces to Parkin's pitch alone. In these terms, one can state that Parkin's pitch is a sufficient condition but it is a special case of the general form given in Eq. (24). With $R = 0$, the screw given in Eq. (21) becomes

$$\mathcal{P}_{1(n,\alpha)} = x \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{\tan(\alpha/2)} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -d \\ 0 \\ 0 \end{bmatrix}. \quad (25)$$

The first three basis screws appearing in Eq. (25) are zero pitch screws, and their axes are confined, respectively, in the coordinate axes OX , OY , OZ . Hence each passing through the origin which is the midpoint of the line segment between v_1 and v_2 . However, the fourth screw is a pure translation screw whose axis is parallel to the line (OX) passing through v_1 and v_2 . These results are in agreement with the results previously obtained in [1].

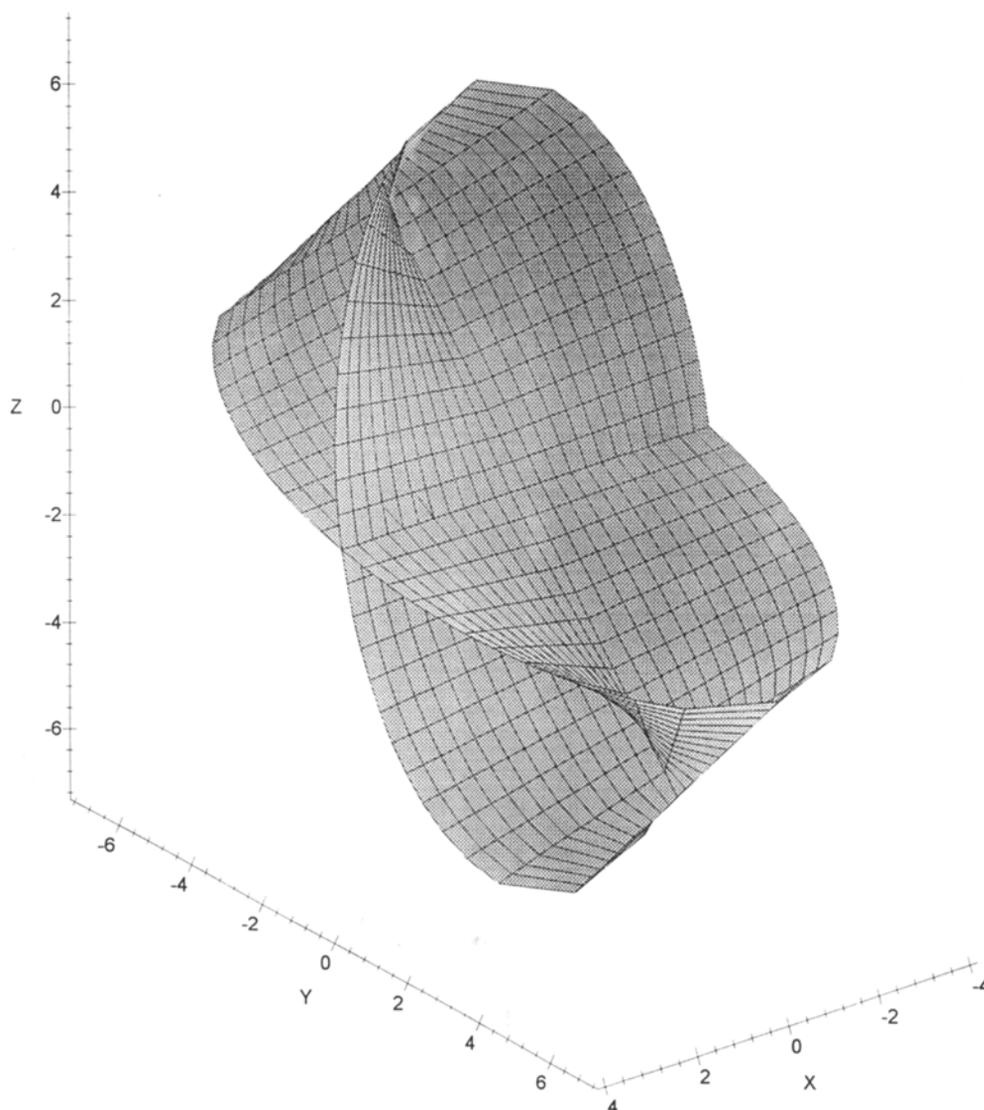


Fig. 5. A plot of W with $d = 5$, $z = y$, where $-1/\sqrt{2} \leq y \leq 1/\sqrt{2}$, and $\alpha = 4\pi/9$

6 Numerical examples

In this section we show that the lines (axes) of screws in the infinity set of screws which is associated with the displacement of a point in a rigid body lie as generators on some ruled surfaces. By making use of Eq. (17), these ruled surfaces W can be parametrized as

$$W(\mathbf{n}, \alpha, t) = \zeta_{1(\mathbf{n}, \alpha)}^\perp + t\mathbf{n}_{1(\mathbf{n}, \alpha)} \quad (26)$$

Where t is a real parameter. Since a single ruled surface can have two parameters, then the following two cases will be considered:

1) The unit direction $\mathbf{n}_{1(\mathbf{n}, \alpha)}$ is taken as fixed by selecting the point B which lies in the interior of the unit circle C in the OYZ plane, while α changes. This case corresponds to case (c)

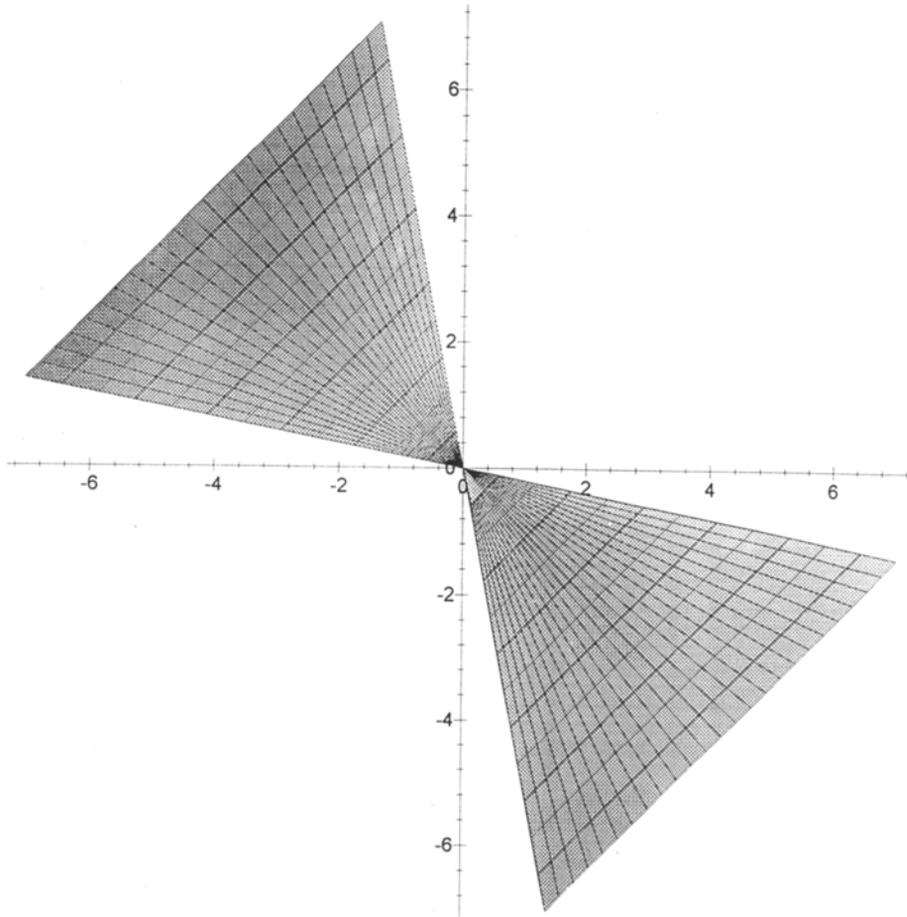


Fig. 6. A plot showing the projection of the ruled surface in Fig. 5 on OYZ plane

in Section 4. As an example, a plot of W with $d = 5, y = 0, z = 1/\sqrt{2}$ and $\pi/9 \leq \alpha \leq \pi/4$ is shown in Fig. 3.

2) The angle α is fixed while z is taken to be the function of y , together with the fact that $x^2 + y^2 + z^2 = 1$. As examples, Fig. 4 shows a plot of W with $d = 4\sqrt{2}, y^2 + z^2 = 1/2$, where $-1/\sqrt{2} \leq y \leq 1/\sqrt{2}$, and $\alpha = \pi/2$. Similarly, Fig. 5 shows a plot of W with $d = 5, z = y$, where $-1/\sqrt{2} \leq y \leq 1/\sqrt{2}$, and $\alpha = 4\pi/9$, while its projections on the OYZ plane and OXY plane are shown, respectively, in Figs. 6 and 7. Note that in Fig. 5 the two central screw axes have 180° angle in between while the two screw axes at both ends have 0° angle in between. This type of surface is known as conoid [2]. It is clear that many other surfaces corresponding to different functions, $z = f(y)$, are also possible.

7 Conclusions

The characterization of the screw which realizes the displacement of a point on a rigid body moving arbitrarily from one position to another in space is described. Since there are ∞^3 such possible screws, using the concept of screw matrix it is shown that the set of these ∞^3 screws form a 4-system if and only if the pitch for the screw is defined properly. In fact, the establish

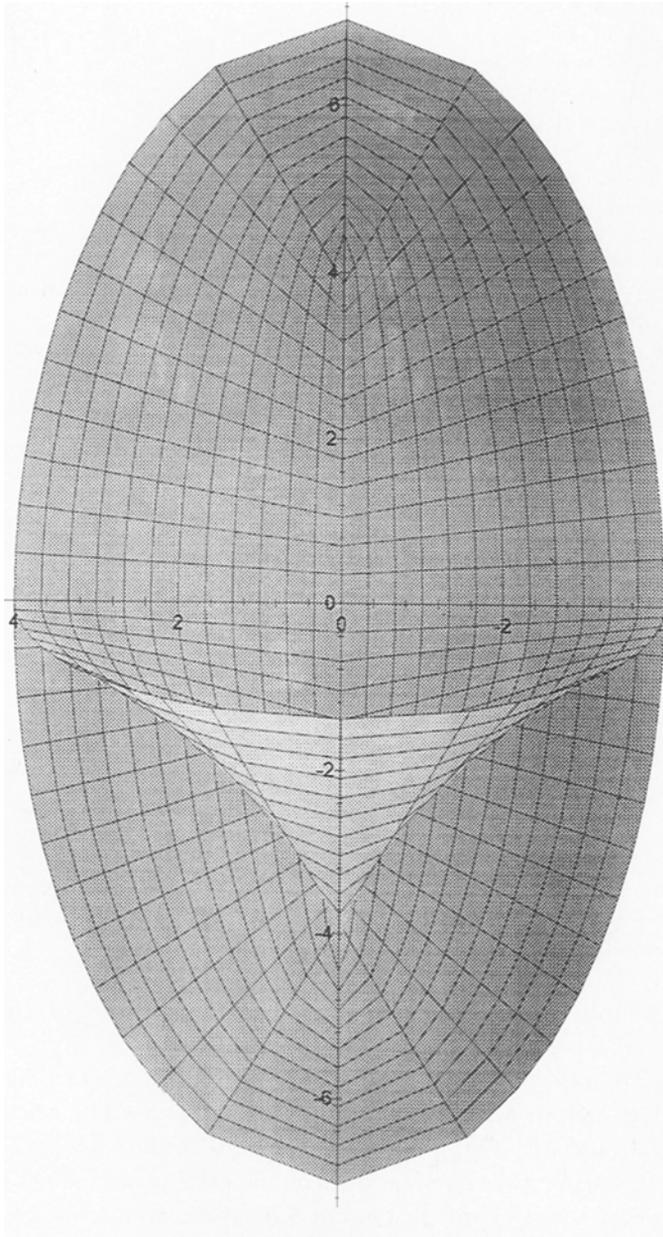


Fig. 7. A plot showing the projection of the ruled surface in Fig. 5 on OXY plane

expression of the pitch includes the pitch defined by Parkin plus an arbitrary constant which may be taken as zero. Moreover, several new geometrical properties of the set of ∞^3 screws are obtained, examples are also given to illustrate some of these properties. Finally, the results in this paper, in addition to their theoretical importance, have application when only the displacement of a point in the rigid body is specified.

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