Meccanica **33:** 111–125, 1998.

On the Finite Screw Cylindroid Represented as a 2-system of Screws

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(Received: 2 June 1997; accepted in revised form: 8 October 1997)

Abstract. The problem of displacing a line with a definite point on it from one spatial position to another is studied by utilizing the concept of screw matrix. It is known that all the available finite twists (screws) associated with this displacement form a ruled surface, the so-called finite screw cylindroid. If the definition of the pitch given by Parkin is used, then the finite screw cylindroid can be regarded as a 2-system of screws. This brings to one's mind the question as to whether there exist different appropriate measures for pitch other than Parkin's under which all the available finite twists form a 2-system. This question is answered in this paper. By deriving a general expression of the pitch for these available finite twists under the said condition, it is shown that Parkin's pitch plus an arbitrary constant is the only possible measure of pitch under which the finite screw cylindroid represents a 2-system of screws. However, since adding a constant to the pitches of all screws of any 2-system still gives a 2-system, constant term may be omitted. It is also shown that the determined 2-system of screws can be described as a linear combination of two special basis screws which are called in this paper the $\alpha = 0$ and the $\alpha = \pi$ screws.

Sommario. Il problema dello spostamento di una linea con un punto definito viene studiato utilizzando il concetto di matrice elicoidale. E' noto che tutte le possibili rototraslazioni elicoidali associate con questo spostamento formano una superficie rigata chiamata cilindroide elicoidale finito. Se viene usata per il passo la definizione di Parkin, allora il cilindroide può essere considerato come un doppio sistema di viti. Ciò pone il problema della possibile esistenza di altre definizioni di passo (oltre a quella di Parkin) per le quali tutte le rototraslazioni elicoidali formino un doppio sistema di viti. In questo lavoro si fornisce una risposta a questa questione. Derivando una espressione generale del passo per le rotazioni elicoidali con le condizioni citate, viene mostrato che il passo di Parkin più una costante arbitraria è l'unica possibile definizione di passo per la quale il cilindroide degli assi elicoidali rappresenti un doppio sistema di viti. Visto che la costante additiva è arbitraria, il termine costante può essere omesso. Viene anche mostrato che il sistema a doppia vite così determinato può essere descritto come una combinazione lineare di due elicoidi elementari che in questo lavoro vengono chiamati l'elicoide $\alpha = 0$ e quello $\alpha = \pi$.

Key words: Screw systems, General mechanics.

1. Introduction

The displacement of a rigid body in space, when completely specified, can be described as the combination of a rotation (angle) about and a translation along a unique screw axis; the ratio of the translation to the rotation is called the cardinal pitch of the finite screw [1]. This displacement is usually referred to as finite twist displacement [2]. The mathematical analog of a physical screw which would produce a specified spatial displacement of a rigid body is the screw matrix. Beggs [3] has given a derivation for a screw matrix by using two different coordinate systems; however, the resulting screw matrix is a complicated 4×4 matrix. In a right-hand coordinate system this matrix has a determinant equals unity, while its inverse is obtained by negating the translation along and the rotation about the corresponding screw axis.

The finite displacement of a rigid body can completely be represented by six independent parameters. When less than six independent parameters are specified, the displacement is called incompletely specified displacement [2]. The displacement of a line with a point on it which is considered as the element of a rigid body is an incompletely specified displacement. Indeed, only five independent parameters are specified rather than six. The transformation associated with this displacement depends upon one free parameter. Hence, there are ∞^1 available screws to perform the same displacement. All of the possible screw axes actually form a ruled surface, the so-called the finite screw cylindroid. Many of the properties related with the cylindroid are given in [1, 4, 5]. Previously, Tsai and Roth [6] had studied all possible screws associated with the incompletely specified displacements of a line with a point, a line only, a point and a point on a plane. They have investigated the problem based on the concept of the screw triangle [1, 7]. Sticher [5], by choosing a special coordinate system, had derived the finite screw cylindroid. He followed a geometric approach different from that in [6]. Parkin [4], by defining the pitch for a finite screw as the ratio of one-half the translation to the tangent of one-half the rotation, has shown that the finite screw cylindroid can be represented by the linear combinations of two bases screws. In other words, Perkin's work shows under his definition of pitch that all available finite screws for displacing a line with a point form a linearly dependent set, namely, a 2-system of screws. Huang and Roth [2] showed that, in addition to the case studied by Parkin [4] and by using Parkin's definition of pitch, the screw systems associated with other incompletely specified displacements can also be represented by linear systems or their nonlinear subsets. They derived explicit analytic expressions for screw systems corresponding to finite displacements of a line with a point, a line only and also a point. Finally, Huang and Chen [8], by using Parkin's definition of pitch, derived the linear representation of the screw triangle. Using this linear representation of the screw triangle, they also performed the finite kinematic analysis of multi-link serial chains to demonstrate a unification of finite and infinitesimal kinematics.

In this paper, using only ordinary linear algebra, the screw matrix as a transformation matrix is expressed in a fixed reference coordinate system. By using this transformation matrix in the study of the incompletely specified displacement of a line with a point, the possibility of obtaining simplified analytic description of the set of the ∞^1 available screws associated with this displacement is given. Without loss of generality, the Sticher's coordinate system is selected to further simplify the re-derivation of the finite screw cylindroid and many of its basic properties. This selection makes it also possible in an easy way to derive a general expression (measure) of the pitches for the finite screw cylindroid. Note that Parkin only defined an appropriate pitch and did not give any clue as to how he arrived at this definition. Although it appears that Parkin's definition is sufficient to represent the cylindroid as a 2-system, it brings to one's mind the question that whether there exists pitches other than Parkin's under which all the available finite twists will also form a 2-system. It is found that Parkin's definition of pitch, plus an arbitrary constant term, is the only suitable measure of pitch. On the other hand, the constant term can be omitted without altering the existence of the 2-system. Therefore one can state that Parkin's pitch is not only a sufficient condition for the cylindroid to represent a 2-system of screws but it is also a necessary condition. Furthermore, it is shown that the determined 2-system of screws can be described as a linear combination of two special screws which are called in this paper the $\alpha = 0$ and the $\alpha = \pi$ screws.

2. Preliminary Considerations

A special vector $\boldsymbol{a}$, in a fixed Cartesian coordinate system Σ_0 is represented by a 3×1 column matrix or a 3×3 skew symmetric matrix:

$$\boldsymbol{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}, \quad \boldsymbol{A} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}, \quad (\boldsymbol{A}^T = -\boldsymbol{A}). \tag{1}$$

Therefore, the scalar product $l = \boldsymbol{a} \cdot \boldsymbol{b}$ and the vector product $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$ of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are conveniently denoted in Σ_o by the respective matrix products $l = \boldsymbol{a}^T \boldsymbol{b}$ and $\boldsymbol{c} = \boldsymbol{A}\boldsymbol{b}$. The use of such representation usually provides some simplifications in the establishment of vectorial relations, see Tokad [9].

The Plucker representation of a line $\boldsymbol{L}$ in space may be given by a pair of column matrices $\boldsymbol{L} = (\boldsymbol{n}; \boldsymbol{m})$, where $\boldsymbol{n}$ and $\boldsymbol{m}$, respectively, represent the direction cosine vector of the line with $\boldsymbol{n}^T \boldsymbol{n} = 1$, and its moment vector with respect to the origin of Σ_o:

$$\boldsymbol{m} = -\boldsymbol{N}\zeta, \tag{2}$$

where ζ is the coordinate matrix of any point on the line. Since $\boldsymbol{N}\boldsymbol{n} = 0$ and $\boldsymbol{n}^T \boldsymbol{N} = 0$, then $\boldsymbol{n}^T \boldsymbol{m} = 0$ is the constraint on the pair $(\boldsymbol{n}; \mathbf{m})$. When the pair of column matrices $\boldsymbol{n}$ and $\boldsymbol{m}$ are given, satisfying the condition $\boldsymbol{n}^T \boldsymbol{m} = 0$, then from equation (2) the position vector can be solved as

$$\zeta = \lambda \boldsymbol{n} + \boldsymbol{N}\boldsymbol{m}, \tag{3}$$

where λ is an arbitrary real number, and the second term $\boldsymbol{N}\boldsymbol{m}$ represents the perpendicular component of ζ with respect to $\boldsymbol{n}$ i.e.,

$$\zeta^{\perp} = \boldsymbol{N}\boldsymbol{m}. \tag{4}$$

A screw $\wp$ is a line of unit direction $\boldsymbol{n}$ at a radius vector $\zeta^{\perp}$ from the origin with an associated scalar pitch value p. Therefore, a line is a screw with zero pitch. A general screw can be represented in Plucker coordinates as

$$\wp = (\boldsymbol{n}^o; \boldsymbol{m}^o) = (\boldsymbol{n}; \boldsymbol{m} + p\boldsymbol{n}). \tag{5}$$

Hence, with this notation, $\zeta^{\perp} = \boldsymbol{N}\boldsymbol{m}^o = \boldsymbol{N}\boldsymbol{m}$ and $p = \boldsymbol{n}^T \boldsymbol{m}^o$.

n-system of screws is a set of screws constructed by the linear combination of n linearly independent (basis) screws [1]. When any two linearly independent screws $\wp_1 = (\boldsymbol{n}_1; \boldsymbol{m}_1 + p_1\boldsymbol{n}_1)$ and $\wp_2 = (\boldsymbol{n}_2; \boldsymbol{m}_2 + p_2\boldsymbol{n}_2)$ are given as basis, the set of all possible real linear combination of those screws form a 2-system of screws. Any screw $\wp$ belonging to this set can be represented as $\wp = (\mu_1\boldsymbol{n}_1 + \mu_2\boldsymbol{n}_2; \mu_1(\boldsymbol{m}_1 + p_1\boldsymbol{n}_1) + \mu_2(\boldsymbol{m}_2 + p_2\boldsymbol{n}_2))$ where μ_1 and μ_2 are two arbitrary real constants.

3. The Screw Matrix

When a new Cartesian coordinate system Σ is attached rigidly to a body and the body rotating about the origin of the fixed coordinate system Σ_o, the determination of the displacement of a point on the body requires a knowledge on the coordinate transformation or rotation matrix $\boldsymbol{T}$ which may be given by [7, 10, 11]

$$\boldsymbol{T} = \cos\theta \boldsymbol{I} + (1 - \cos\theta)\boldsymbol{n}\boldsymbol{n}^T + \sin\theta \boldsymbol{N} = \boldsymbol{I} + (1 - \cos\theta)\boldsymbol{N}^2 + \sin\theta \boldsymbol{N}, \tag{6}$$

where, according to Euler's theorem [12], $\boldsymbol{n}$ is the direction cosine vector of the rotation axis and the θ is the rotation angle about this axis. $\boldsymbol{T}$ is an orthonormal matrix and $(\boldsymbol{I} - \boldsymbol{T})\boldsymbol{n} = 0$. When $\boldsymbol{T}$ is given one can obtain the parameters θ and $\boldsymbol{n}$ from

$$\cos\theta = \frac{1}{2}(1 - \text{tr } \boldsymbol{T}); \quad \boldsymbol{N} = \frac{1}{2\sin\theta}(\boldsymbol{T} - \boldsymbol{T}^T), \tag{7}$$

where tr(.) indicates the trace of the square matrix (.).

On the other hand, if Σ is not only rotating but also translating by an amount ζ with respect to Σ_o, then the homogeneous transformation matrix [13, 14] of order four is required for the determination of the general displacement:

$$\boldsymbol{H} = \begin{bmatrix} \boldsymbol{T} & \xi \\ 0 & 1 \end{bmatrix}.$$

However, by Chasles theorem [12], the same displacement can be obtained by a finite twist about a certain screw axis. To find a mathematical representation of this equivalent finite twist, we again let $\boldsymbol{n}$ be the direction cosine matrix of the screw axis and ζ be the coordinate matrix of a point on this screw axis. The finite twist displacement can be achieved in four steps. First translate the screw axis to the origin of Σ_o, then rotate the moving coordinate system Σ by an angle θ about this new position of the screw axis. Then translate the screw axis back to its initial location, and finally translate Σ by an amount λ along the screw axis. These operations can be expressed in terms of homogeneous transformation matrices yielding

$$\begin{bmatrix} \boldsymbol{I} & \lambda\boldsymbol{n} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & \zeta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{T}^* & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & -\zeta \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{T} & \xi \\ 0 & 1 \end{bmatrix}, \tag{8}$$

where $\boldsymbol{T}^*$ is a 3×3 orthonormal matrix representing the rotation about the screw axis shifted to the origin. After simplification, equation (8) becomes

$$\$ = \begin{bmatrix} \boldsymbol{T}^* & (\boldsymbol{I} - \boldsymbol{T}^*)\zeta + \lambda\boldsymbol{n} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{T} & \xi \\ 0 & 1 \end{bmatrix}. \tag{9}$$

The 4×4 matrix \$ on the left-hand side of equation (9) is the screw matrix in its compact form. When $\boldsymbol{T}$ and ξ are given, then $\boldsymbol{T}^* = \boldsymbol{T}$ and

$$(\boldsymbol{I} - \boldsymbol{T}^*)\zeta + \lambda\boldsymbol{n} = -((1 - \cos\theta)\boldsymbol{N}^2 + \sin\theta\boldsymbol{N})\zeta + \lambda\boldsymbol{n} = \xi. \tag{10}$$

Now, from equation (7) we know the direction of the screw axis, and the rotation about it. We also have, from equation (10), the translation along the screw axis as $\lambda = \boldsymbol{n}^T\xi$. To determine the location of the screw axis, i.e., ζ, the relation in equation (10) can not be used directly since $(\boldsymbol{I} - \boldsymbol{T})$ is a singular matrix. However, one can show that these equations are consistent and yield infinitely many solutions for ζ. This is expected since the vector ζ describes the screw axis. In the following we follow a different rout for the solution. One can modify the relation in equation (10) into

$$((1 - \cos\theta)\boldsymbol{N} + \sin\theta\boldsymbol{I})\boldsymbol{m} = -\boldsymbol{N}^2\xi.$$

Since the coefficient matrix of the moment vector $\boldsymbol{m}$ is now nonsingular with the inverse

$$[(1 - \cos\theta)\boldsymbol{N} + \sin\theta\boldsymbol{I}]^{-1} = \frac{1}{2}\left[\frac{1}{\tan(\theta/2)}\boldsymbol{I} + \tan(\theta/2)\boldsymbol{n}\boldsymbol{n}^T - \boldsymbol{N}\right]$$

then the moment $\boldsymbol{m}$ of the screw axis, with respect to the origin of Σ_o, is

$$\boldsymbol{m} = -\frac{1}{2}\left[\frac{1}{\tan(\theta/2)}\boldsymbol{N} + \boldsymbol{I}\right]\boldsymbol{N}\xi. \tag{11}$$

Furthermore the perpendicular vector from the origin of Σ_o to the screw axis becomes

$$\zeta^{\perp} = \frac{1}{2}\left[\frac{1}{\tan(\theta/2)}\boldsymbol{I} - \boldsymbol{N}\right]\boldsymbol{N}\xi. \tag{12}$$

Therefore, the physical properties of a screw can be summarized in a quadruple of the form $(\theta, \boldsymbol{n}, \lambda, \zeta^{\perp})$.

Note also that the screw matrix \$ has a determinant equals unity and an inverse

$$\$^{-1} = \begin{bmatrix} \boldsymbol{T}^T & (\boldsymbol{I} - \boldsymbol{T}^T)\zeta - \lambda\boldsymbol{n} \\ 0 & 1 \end{bmatrix}.$$

It is also not difficult to show that the expression of \$ described in Σ coordinate system is

$$\$_{\Sigma} = \boldsymbol{H}^{-1}\$\boldsymbol{H} = \begin{bmatrix} \boldsymbol{T} & (\boldsymbol{I} - \boldsymbol{T}^T)(\xi - \zeta) + \mathbf{T}^T\lambda\boldsymbol{n} \\ 0 & 1 \end{bmatrix}.$$

Successive finite twist displacements can be represented as the product of screw matrices. For instance, the product $\$_3 = \$_2\$_1$ suggests that there is a single screw $\wp_3$, equivalent to two corresponding screws $\wp_1$ and $\wp_2$ which are operated on Σ coordinate in this order. Therefore the physical properties of $\wp_3$ can uniquely be determined in terms of those of $\wp_1$ and $\wp_2$.

4. General Displacement of a Directed Line with a Point on it

We consider two oriented lines, L_1 and L_2, with the corresonding direction cosine vectors $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ respectively, in Σ_o together with respective points $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ on these lines. The position vectors of the points $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are respectively $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. As it is mentioned in the introduction, a transformation which carries $\boldsymbol{g}_1$ into $\boldsymbol{g}_2$ and $\boldsymbol{v}_1$ into $\boldsymbol{v}_2$ is not unique. Therefore there are ∞^1 set of screws that can perform the same transformation. We are interested in describing this set. To obtain analytic expressions for describing all possible screws, the displacement can be decomposed into two steps: first, transform $\boldsymbol{g}_1$ into $\boldsymbol{g}_2$ and $\boldsymbol{v}_1$ into $\boldsymbol{v}_2$. Second, rotate about L_2 by an amount α, $(-\pi < \alpha < \pi)$, where α is the free parameter is this displacement. The first step can be performed by a homogeneous matrix $\boldsymbol{H}_{12}$ whose submatrices $\boldsymbol{T}_{12}$ and ξ_{12} are completely determined as

$$\begin{aligned} \boldsymbol{T}_{12} &= \boldsymbol{I} + (1 - \cos\theta_{12})\boldsymbol{N}_{12}^2 + \sin\theta_{12}\boldsymbol{N}_{12}; \\ \xi_{12} &= \boldsymbol{v}_{12} - \boldsymbol{T}_{12}\boldsymbol{v}_1 \end{aligned} \tag{13}$$

in which

$$\boldsymbol{n}_{12} = \frac{\boldsymbol{G}_1\boldsymbol{g}_2}{\mid \boldsymbol{G}_1\boldsymbol{g}_2 \mid} \quad \text{and} \quad \tan\theta_{12} = \frac{\mid \boldsymbol{G}_1\boldsymbol{g}_2 \mid}{\mid \boldsymbol{g}_1^T\boldsymbol{g}_2 \mid}.$$

Note that $\boldsymbol{N}_{12}$ is the skew symmetric representation of the common perpendicular, $\boldsymbol{n}_{12}$, of the vectors $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$, while θ_{12} is the angle between $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ measured, in a right hand rule, about the common perpendicular $\boldsymbol{n}_{12}$. The second step can be performed by a screw matrix $\$_{2\alpha}$ which represents a zero translation screw whose respective axis and angle of rotation are L_2 and α.

However, the entire displacement from location 1 to any location determined by the parameter α, can be described by a single screw matrix $\$_{1\alpha}$. This operation can be represented as

$$\begin{aligned}\$_{1\alpha} = \$_{2\alpha}\boldsymbol{H}_{12} &= \begin{bmatrix} \boldsymbol{T}_{2\alpha} & (\boldsymbol{I}-\boldsymbol{T}_{2\alpha})\boldsymbol{v}_2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \boldsymbol{T}_{12} & \xi_{12} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{T}_{1\alpha} & (\boldsymbol{I}-\boldsymbol{T}_{1\alpha}\zeta_{1\alpha}+\lambda_{1\alpha}\boldsymbol{n}_{1\alpha} \\ 0 & 1 \end{bmatrix},\end{aligned} \tag{14}$$

where

$$\boldsymbol{T}_{2\alpha} = \boldsymbol{I} + (1-\cos\alpha)\boldsymbol{G}_2^2 + \sin\alpha\boldsymbol{G}_2. \tag{15}$$

From equation (14) and by making use of equations (7) and (12), the physical properties of the equivalent screws $(\theta_{1\alpha}, \boldsymbol{n}_{1\alpha}, \lambda_{1\alpha}, \zeta_{1\alpha}^{\perp})$ are

$$\begin{aligned}\cos\theta_{1\alpha} &= \frac{1}{2}(1-\operatorname{tr}\boldsymbol{T}_{1\alpha});\ \boldsymbol{N}_{1\alpha} = \frac{1}{2\sin\theta_{1\alpha}}(\boldsymbol{T}_{1\alpha}-\boldsymbol{T}_{1\alpha}^T); \\ \lambda_{1\alpha} &= \boldsymbol{n}_{1\alpha}^T\xi_{1\alpha};\ \zeta_{1\alpha}^{\perp} = \frac{1}{2}\left[\frac{1}{\tan(\theta_{1\alpha}/2)}\boldsymbol{I}-\boldsymbol{N}_{1\alpha}\right]\boldsymbol{N}_{1\alpha}\xi_{1\alpha},\end{aligned} \tag{16}$$

where

$$\boldsymbol{T}_{1\alpha} = \boldsymbol{T}_{2\alpha}\boldsymbol{T}_{12} \quad \text{and} \quad \xi_{1\alpha} = \boldsymbol{v}_2 - \boldsymbol{T}_{1\alpha}\boldsymbol{v}_1.$$

Equation (16) gives analytic description of all possible screws associated with the general spatial displacement of a line with a point on it. For the simplification of this analytic description, the Sticher's selection of coordinate system is specially considered in the following section.

5. The Finite Screw Axis Cylindroid

Figure 1 displays the Sticher's selection of coordinate system in which the two locations (L_1 and L_2) of the line are brought to specific positions. In this selection let $\mathbf{v}_1$ and $\mathbf{v}_2$ be located, respectively, at distances q and $q+h$ from the x-axis, then the following expressions can be written

$$\begin{aligned}\boldsymbol{g}_1 &= [\,0\ \ \sin\beta\ \ \cos\beta\,]^T, \quad \boldsymbol{g}_2 = [\,0\ \ -\sin\beta\ \ \cos\beta\,]^T \\ \boldsymbol{v}_1 &= [\,d\ \ q\sin\beta\ \ q\cos\beta\,]^T, \quad \boldsymbol{v}_2 = [\,-d\ \ -(q+h)\sin\beta\ \ (q+h)\cos\beta\,]^T.\end{aligned}$$

By making use of equation (13), we obtain

$$\boldsymbol{T}_{12} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 2\beta & -\sin 2\beta \\ 0 & \sin 2\beta & \cos 2\beta \end{bmatrix}; \quad \xi_{12} = [\,-2d\ \ -h\sin\beta\ \ h\cos\beta\,]^T \tag{17}$$

and from equation (15) we have

$$\boldsymbol{T}_{2\alpha} = \begin{bmatrix} \cos\alpha & -\sin\alpha\cos\beta & -\sin\alpha\sin\beta \\ \sin\alpha\cos\beta & \cos\alpha\cos^2\beta+\sin^2\beta & (\cos\alpha-1)\cos\beta\sin\beta \\ \sin\alpha\sin\beta & (\cos\alpha-1)\cos\sin\beta & \cos\alpha\sin^2\beta+\cos^2\beta \end{bmatrix}. \tag{18}$$

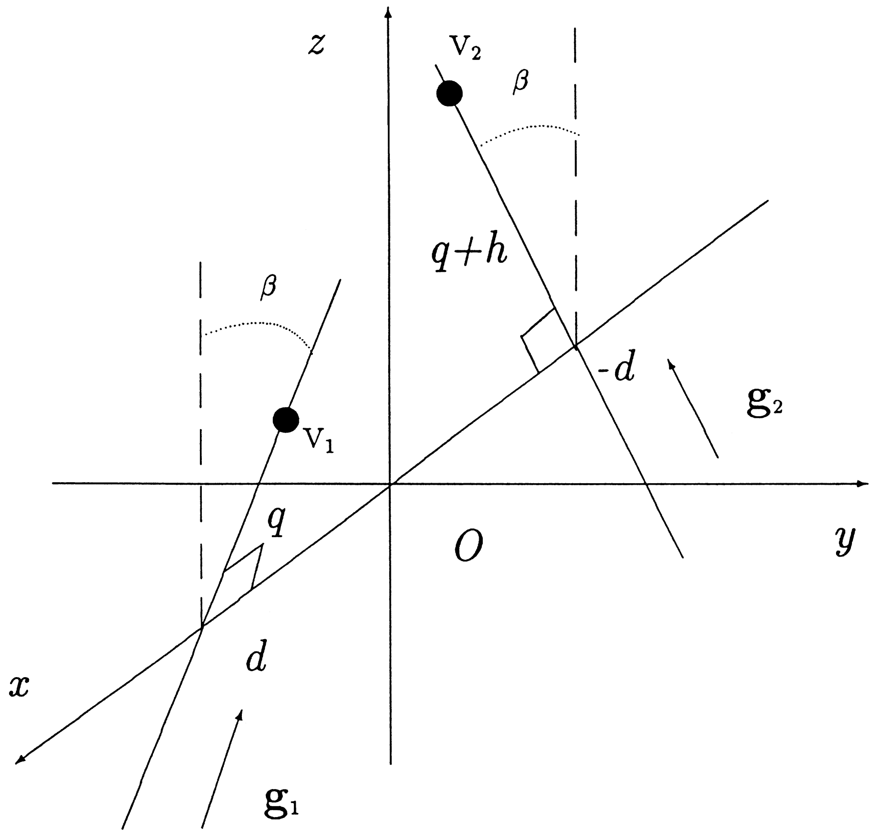

Figure 1. The Sticher's coordinate system.

Hence, using equation (16), the physical properties of the equivalent screw are

$$\cos\theta_{1\alpha} = -\tfrac{1}{2}(\cos 2\beta + (1+\cos 2\beta)\cos\alpha - 1)$$

or

$$\begin{aligned}
\tan\left(\frac{\theta_{1\alpha}}{2}\right) &= \frac{1}{\cos\beta}\sqrt{\sin^2\beta + \tan^2(\alpha/2)};\\
\boldsymbol{n}_{1\alpha} &= \frac{\sin\beta\cos\beta(\cos\alpha+1)}{\sin\theta_{1\alpha}}\begin{bmatrix} 1 & 0 & \dfrac{\tan(\alpha/2)}{\sin\beta}\end{bmatrix}^T\\
&= \frac{\sin\beta}{\sqrt{\sin^2\beta+\tan^2(\alpha/2)}}\begin{bmatrix} 1 & 0 & \dfrac{\tan(\alpha/2)}{\sin\beta}\end{bmatrix}^T;\\
\lambda_{1\alpha} &= \frac{\cos\beta}{\sin\theta_{1\alpha}}(h\cos\beta\sin\alpha - 2d\sin\beta(1+\cos\alpha))\\
&= \frac{h\cos\beta\tan(\alpha/2) - 2d\sin\beta}{\sqrt{\sin^2\beta+\tan^2(\alpha/2)}};
\end{aligned}$$

$$\begin{aligned}\zeta_{1\alpha}^{\perp} &= -\left[0 \frac{(d\cos\beta\sin\alpha + h\sin\beta)}{(1+\sin^2\beta) - \cos\alpha\cos^2\beta} 0\right]^T \\ &= -\left[0 \frac{2d\cos\beta\tan(\alpha/2) + h\sin\beta(1+\tan^2(\alpha/2))}{2(\sin^2\beta + \tan^2(\alpha/2))} 0\right]^T, \end{aligned} \tag{19}$$

where $\boldsymbol{n}_{1\alpha}$ and $\zeta_{1\alpha}^{\perp}$ are respectively the unit direction and the location, along the y-axis, of every possible screw axis. In these terms, the two vectors $\boldsymbol{n}_{1\alpha}$ and $\zeta_{1\alpha}^{\perp}$ describe a ruled surface of the type known as cylindroid (or conoid). The nodal line of this cylindroid is contained in the y-axis. Indeed, the expression of $\boldsymbol{n}_{1\alpha}$ in equation (19) shows clearly that all possible screw axes are perpendicular intersectors of the y-axis with the positive slope e in the x–z plane:

$$e = \frac{\tan(\alpha/2)}{\sin\beta}. \tag{20}$$

The expression of $\zeta_{1\alpha}^{\perp}$ in equation (19) suggests that the location of an intersection point along the nodal line is

$$-Y_\alpha = \frac{2d\cos\beta\tan(\alpha/2) + h\sin\beta(1+\tan^2(\alpha/2))}{2(\sin^2\beta + \tan^2(\alpha/2))}. \tag{21}$$

The two end positions of the cylindroid along the nodal line can be found by calculating the derivative of equation (21) with respect to parameter α and equating it to zero. Doing so, the following two solutions for α are found

$$\tan_{1,2}(\alpha/2) = \frac{-h\sin\beta\cos\beta \mp \sin\beta\sqrt{h^2\cos^2\beta + 4d^2}}{2d}. \tag{22}$$

Substituting, $\tan_{1,2}(\alpha/2)$, into equation (21) yields

$$\begin{aligned} Y_1 &= -\frac{h\sin\beta}{2} - \frac{d^2\cos\beta}{\sin\beta(\sqrt{h^2\cos^2\beta + 4d^2} - h\cos\beta)}, \\ Y_2 &= -\frac{h\sin\beta}{2} + \frac{d^2\cos\beta}{\sin\beta(\sqrt{h^2\cos^2\beta + 4d^2} + h\cos\beta)}, \end{aligned}$$

where Y_1 and Y_2 are the two end positions of the cylindroid along the y-axis. Hence, the region of the cylindroid between the two end positions can be described as

$$Y_1 \leqslant Y_\alpha \leqslant Y_2$$

and its mid point is

$$Y_{\text{mid}} = \frac{Y_1 + Y_2}{2} = \frac{-h(\sin^2\beta + 1)}{4\sin\beta}.$$

From equation (21), for any value of $Y_\alpha = Y$, with $Y_1 \leqslant Y \leqslant Y_2$ we have two solutions for tan $(\alpha/2)$

$$\tan_{1,2}(\alpha/2) = \frac{-2d\cos\beta \mp \sqrt{4d^2\cos^2\beta - 4(h\sin\beta + 2Y)(h\sin\beta + 2Y\sin^2\beta)}}{2(h\sin\beta + 2Y)}. \quad (23)$$

Note that the expression under the square root is greater than or equal zero for $Y_1 \leqslant Y \leqslant Y_2$. It follows from equation (23) that for any given value of $Y_\alpha = Y$ in this interval there are exactly two screw axes intersect one another. The slopes of these axes, as given in equation (21), are

$$e_1 = \frac{\tan_1(\alpha/2)}{\sin\beta}, \quad e_2 = \frac{\tan_2(\alpha/2)}{\sin\beta}.$$

Now if we let $Y = Y_{\mathrm{mid}}$, then from equation (23) we get

$$e_{1\mathrm{mid}} = \frac{2d + \sqrt{4d^2 + h^2\cos^2\beta}}{h\cos\beta}, \; e_{2\mathrm{mid}} = \frac{2d - \sqrt{4d^2 + h^2\cos^2\beta}}{h\cos\beta},$$

where $e_{1\mathrm{mid}}$ and $e_{2\mathrm{mid}}$ are, respectively, the slopes of the two central generators. It can be seen easily that the multiplication of these slopes yields $e_{1\mathrm{mid}}e_{2\mathrm{mid}} = -1$. This implies that the two intersecting generators at the center point of the cylindroid are mutually perpendicular.

6. The Cylindroid as a 2-System of Screws

In this section we derive a general expression of pitch under which the cylindroid can be represented as a 2-system of screws. A general screw in the cylindroid can be written in Plucker coordinate as $\wp_{1\alpha} = (\boldsymbol{n}_{1\alpha}; \boldsymbol{m}_{1\alpha} + p_{1\alpha}\boldsymbol{n}_{1\alpha})$. By making use of equations (2)–(4), $\wp_{1\alpha}$ can be written as

$$\wp_{1\alpha} = (\boldsymbol{n}_{1\alpha}; -\boldsymbol{N}_{1\alpha}\zeta_{1\alpha}^{\perp} + p_{1\alpha}\boldsymbol{n}_{1\alpha}) \quad (24)$$

substitution of the expressions of $\boldsymbol{n}_{1\alpha}$ and $\zeta_{1\alpha}^{\perp}$ from equation (19) into (24), together with the fact that $\boldsymbol{N}_{1\alpha}$ is the skew symmetric representation of $\boldsymbol{n}_{1\alpha}$, gives

$$\begin{aligned}\wp = {} & \frac{1}{\sqrt{\sin^2\beta + \tan^2(\alpha/2)}} \times \\ & \times \left(\begin{bmatrix} \sin\beta \\ 0 \\ \tan\left(\frac{\alpha}{2}\right) \end{bmatrix} ; \frac{2d\cos\beta s\tan\left(\frac{\alpha}{2}\right) + h\sin\beta\left(1 + \tan^2\left(\frac{\alpha}{2}\right)\right)}{2(\sin^2\beta + \tan^2(\alpha/2))} \times \right. \\ & \left. \times \begin{bmatrix} -\tan\left(\frac{\alpha}{2}\right) \\ 0 \\ \sin\beta \end{bmatrix} + p_{1\alpha} \begin{bmatrix} \sin\beta \\ 0 \\ \tan\left(\frac{\alpha}{2}\right) \end{bmatrix} \right).\end{aligned}$$

As given in equation (20), the above equation can be expressed in terms of the slope parameters of the screw axis as

$$\wp_{1\alpha} = \frac{1}{\sqrt{1+e^2}} \left(\begin{bmatrix} 1 \\ 0 \\ e \end{bmatrix}; \frac{e^2 h \sin^2\beta + 2ed\cos\beta + h}{2\sin\beta(1+e^2)} \begin{bmatrix} -e \\ 0 \\ 1 \end{bmatrix} + p_{1\alpha} \begin{bmatrix} 1 \\ 0 \\ e \end{bmatrix} \right). \tag{25}$$

For the sake of further simplification, let

$$k_{1\alpha} = \frac{e^2 h \sin^2\beta + 2ed\cos\beta + h}{2\sin\beta(1+e^2)}, \tag{26}$$

therefore, the screw $\wp_{1\alpha}$ takes the simple form

$$\wp_{1\alpha} = \frac{1}{\sqrt{1+e^2}} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + e \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \begin{bmatrix} -ek_{1\alpha} + p_{1\alpha} \\ 0 \\ k_{1\alpha} + ep_{1\alpha} \end{bmatrix} \right). \tag{27}$$

Now, in order that the screw in equation (27), to form a 2-system of screws, it must be put into the form [2]

$$\wp_{1\alpha} = f_{A\alpha}\wp_A + f_{B\alpha}\wp_B, \tag{28}$$

where $f_{A\alpha}$ and $f_{B\alpha}$ are scalar functions of the free parameter α, while $\wp_A$ and $\wp_B$ are two screws in Plucker's representation whose components are functions of known constants. One can easily see that for the screw in equation (27) to be written in the form of equation (28), it is necessary that its moment component could be expressed as

$$\begin{bmatrix} -ek_{1\alpha} + p_{1\alpha} \\ 0 \\ k_{1\alpha} + ep_{1\alpha} \end{bmatrix} = \begin{bmatrix} R_1 \\ 0 \\ R_2 \end{bmatrix} + e \begin{bmatrix} R_3 \\ 0 \\ R_4 \end{bmatrix}, \tag{29}$$

where R_i, $i = 1, 2, 3, 4$ are real constants to be determined. Actually, from equation (29) the following two relations can be written as

$$p_{1\alpha} = \frac{R_2 - k_{1\alpha}}{e} + R_4, \tag{30}$$

$$k_{1\alpha} = \frac{2\sin\beta(R_2 + e(R_4 - R_1) - e^2 R_3)}{2\sin\beta(1+e^2)}. \tag{31}$$

Comparing the expression of $k_{1\alpha}$ in equation (31) with that in equation (26), the following relation can be obtained

$$R_1 = R_4 - \frac{d\cos\beta}{\sin\beta}; \quad R_2 = \frac{h}{2\sin\beta}; \quad R_3 = -\frac{h\sin\beta}{2},$$

in which R_4 remains as an arbitrary constant. By direct substitution of R_2 and $k_{1\alpha}$ (in equation (26)) into equation (30), a general expression of pitch for the 2-system associated with the finite screw cylindroid is obtained as

$$p_{1\alpha} = \frac{eh\cos\beta - 2d}{2(1+e^2)\tan\beta} + R_4. \tag{32}$$

Comparing this expression with that in equation (19), we finally have

$$p_{1\alpha} = \frac{\lambda_{1\alpha}}{2\tan(\theta_{1\alpha}/2)} + R_4. \tag{33}$$

The first term of the right-hand side of equation (33) is Parkin's definition of pitch. Therefore, Parkin's pitch, plus an arbitrary real constant, is the only pitch under which the finite screw axis cylindroid can be represented as a 2-system of screws. In the example section we show that R_4 can take any real value without changing the physical properties of the general screw $\wp_{1\alpha}$. Therefore, setting $R_4 = 0$ will always be adequate. In fact, Parkin in his work [4] made use of this property by a uniform additive adjustment to the pitch of every generator belonging to the finite screw cylindroid.

7. Analytic Description of the Determined 2-System

In this section we show that the determined 2-system can be described as a linear combination of two special screws. If we consider the expressions of R_i, $i = 1, 2, 3, 4$ the general screw $\wp_{1\alpha}$ can be written as

$$\wp_{1\alpha} = \frac{1}{\sqrt{1+e^2}}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}; \begin{bmatrix} R_4 - \dfrac{d}{\tan\beta} \\ 0 \\ \dfrac{h}{2\sin\beta}\end{bmatrix}\right) + \frac{e}{\sqrt{1+e^2}}\left(\begin{bmatrix}0\\0\\1\end{bmatrix}; \begin{bmatrix} -\dfrac{h\sin\beta}{2} \\ 0 \\ R_4\end{bmatrix}\right). \tag{34}$$

Comparing equation (34) with that in equation (28), the screws $\wp_A$ and $\wp_B$ are:

$$\wp_A = \left(\begin{bmatrix}1\\0\\0\end{bmatrix}; \begin{bmatrix} R_4 - \dfrac{d}{\tan\beta} \\ 0 \\ \dfrac{h}{2\sin\beta}\end{bmatrix}\right) \quad \text{and} \quad \wp_B = \left(\begin{bmatrix}0\\0\\1\end{bmatrix}; \begin{bmatrix} -\dfrac{h\sin\beta}{2} \\ 0 \\ R_4\end{bmatrix}\right).$$

One can easily see from equation (34) that the two screws $\wp_A$ and $\wp_B$ are special screws corresponding, respectively, to $\alpha = 0$ (i.e., $e = 0$), and (in the limiting case) $\alpha = \pi$ (i.e., $e \to \infty$). Therefore, it is reasonable to call these two special screws, respectively, as $\wp_{\alpha=0}$ and $\wp_{\alpha=\pi}$ screws. The unit direction of $\wp_{\alpha=\pi}$ is $\boldsymbol{n}_{\alpha=\pi} = \boldsymbol{G}_1\boldsymbol{g}_2/|\boldsymbol{G}_1\boldsymbol{g}_2|$ and has a pitch value $p_{\alpha=\pi} = R_4$ and intersecting the nodal line (the y-axis) at $Y_{\alpha=\pi} = -(h\sin\beta)/2$. However, the unit direction, pitch value and the intersection point along the nodal line of $\wp_{\alpha=0}$ are, respectively, $\boldsymbol{n}_{\alpha=0} = (\boldsymbol{g}_1 + \boldsymbol{g}_2)/|\boldsymbol{g}_1 + \boldsymbol{g}_2|$, $p_{\alpha=0} = R_4 - (d/\tan\beta)$ and $Y_{\alpha=0} = -h/(2\sin\beta)$. It is now clear that these two special screws are perpendicular to each other and symmetrically intersecting the nodal line at a distance $|Y| = h\cos^2\beta/4\sin\beta$ measured from the mid point, $Y_{\text{mid}} = -h(\sin^2\beta + 1)/4\sin\beta$, of the cylindroid. In fact, with $R_4 = 0$, these two special screws, namely, $\wp_{\alpha=0}$ and $\wp_{\alpha=\pi}$, correspond, respectively, to Parkin's T^x and T^π key generators [4]. Now, we show how the description in equation (34) of the determined 2-system can directly be established from the unit directions $(\boldsymbol{g}_1, \boldsymbol{g}_2)$ of the given lines and the position vectors $(\boldsymbol{v}_1, \boldsymbol{v}_2)$ of the corresponding points on them. Since the form in equation (34) corresponds to the Sticher's selection of coordinate system with the x-axis along the common perpendicular of $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$, then all what is needed to evaluate equation (34) is to determine the values of the constants d, h and β. In fact these constants can be calculated directly as

$$d = (\boldsymbol{v}_1 - \boldsymbol{v}_2)^{\mathrm{T}} \frac{\boldsymbol{G}_1 \boldsymbol{g}_2}{2 \mid \boldsymbol{G}_1 \boldsymbol{g}_2 \mid} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \end{bmatrix} (\boldsymbol{v}_1 - \boldsymbol{v}_2);\ h = \boldsymbol{v}_2^{\mathrm{T}} \boldsymbol{g}_2 - \boldsymbol{v}_1^{\mathrm{T}} \boldsymbol{g}_1$$

$$\tan 2\beta = \frac{\mid \boldsymbol{G}_1 \boldsymbol{g}_2 \mid}{\mid \boldsymbol{g}_1^{\mathrm{T}} \boldsymbol{g}_2 \mid}. \tag{35}$$

In what follows we show that a resultant screw of any real combination of two linearly independent screws which belong to the finite screw cylindroid is also a screw belonging to the same cylindroid (see also [2]). Now we may let $\wp_{11}$ and $\wp_{12}$ be any two linearly independent screws obtained by using equation (34) corresponding to two different free parameters α_1 and α_2, then the linear combination of these two screws is $\wp_{13} = \mu_1 \wp_{11} + \mu_2 \wp_{12}$, where μ_1 and μ_2 are arbitrary real constants. In these terms the direction component (not necessarily normalized) of $\wp_{13}$ has the form

$$\boldsymbol{n}_{13} = \frac{\mu_1}{\sqrt{1+e_1^2}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{e_1 \mu_1}{\sqrt{1+e_1^2}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \frac{\mu}{\sqrt{1+e_2^2}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{e_2 \mu_2}{\sqrt{1+e_2^2}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\mu_1}{\sqrt{1+e_1^2}} + \dfrac{\mu_2}{\sqrt{1+e_2^2}} \\ 0 \\ \dfrac{e_1 \mu_1}{\sqrt{1+e_1^2}} + \dfrac{e_2 \mu_2}{\sqrt{1+e_2^2}} \end{bmatrix}. \tag{36}$$

In order that $\boldsymbol{n}_{13}$ in equation (36) to have the same form as that in equation (34), one can see without difficulty that the slope of the axis of the screw $\wp_{13}$ must have the following form

$$e_3 = \frac{\mu_1 e_1 \sqrt{1+e_2^2} + \mu_2 e_2 \sqrt{1+e_1^2}}{\mu_1 \sqrt{1+e_2^2} + \mu_2 \sqrt{1+e_1^2}}. \tag{37}$$

Therefore, for any given μ_1 and μ_2 we obtain a unique screw, belonging to the same cylindroid whose slope is as given in equation (37), and its physical properties $(\theta_{13}, \lambda_{13}, \boldsymbol{n}_{13}, \zeta_{13}^{\perp})$, can be determined directly as demonstrated in the next section.

8. Numerical Examples

In this section we give two numerical examples. In the first example we simply obtain a plot of the finite screw cylindroid. However, in the second example we show how the physical properties of a resultant screw (of any real combination of two linearly independent screws belong to the cylindroid) can directly be determined. We also show that these physical properties are independent of the value of the constant pitch term R_4.

9. Example 1

By making use of equation (19) or equation (34), the finite screw cylindroid (*FSC*) can be parameterized as

$$FSC(\alpha, t) = \zeta_{1\alpha}^{\perp} + t\boldsymbol{n}_{1\alpha}, \tag{38}$$

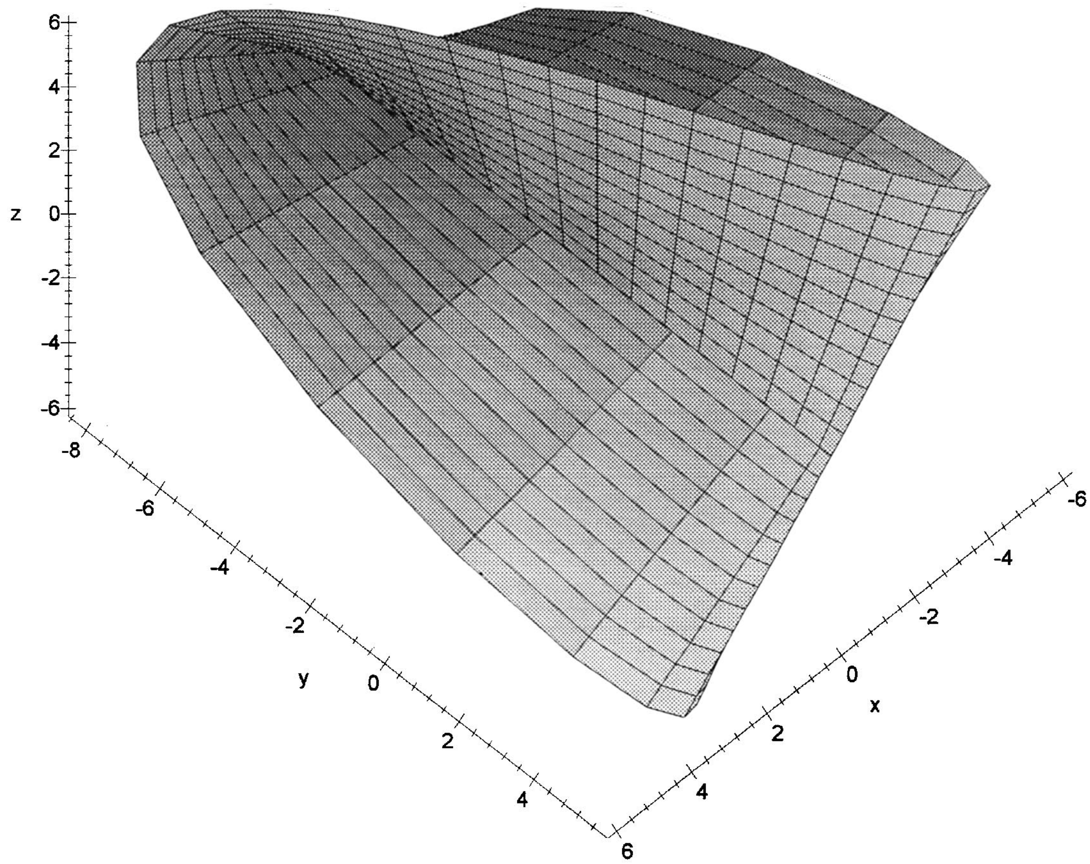

Figure 2. A 3-D, plot of the finite screw cylindroid ($d = 8$, $h = 2$ and $\beta = \pi/3$).

where t is a real number. Let the lines L_1 and L_2 and the points on them be given as

$$\boldsymbol{g}_1 = \begin{bmatrix} 0 \\ 0.86602 \\ 0.5 \end{bmatrix}, \quad \boldsymbol{g}_2 = \begin{bmatrix} 0 \\ -0.86602 \\ 0.5 \end{bmatrix}, \quad \boldsymbol{v}_1 = \begin{bmatrix} 8 \\ 1.73205 \\ 1 \end{bmatrix}$$

$$\text{and} \quad \boldsymbol{v}_2 = \begin{bmatrix} -8 \\ -3.4641 \\ 2 \end{bmatrix},$$

so that from equation (35) we have

$$d = 8, h = 2 \quad \text{and} \quad \beta = \pi/3.$$

Based on these values of d, h and β together with $(\pi - 0.001) \geqslant \alpha \geqslant (-\pi + 0.001)$ and $6 \geqslant t \geqslant -6$, a plot of equation (38) is shown in Figure 2.

10. Example 2

In equation (37), if we let $e_1 = 1$ (i.e., $\alpha_1 = 1.42744$ rad), $e_2 = 4$ (i.e.; $\alpha_2 = 2.57952$ rad), $\mu_1 = 5$ and $\mu_2 = 6.5$, then $e_3 = 1.92516$ (i.e; $\alpha_3 = 2.06105$ rad), and hence from equation (34), with d, h and β being the same as in example 1, we obtain

$$\wp_{13} = 0.4605 \left(\begin{bmatrix} 1 \\ 0 \\ 1.92516 \end{bmatrix} ; \quad \begin{bmatrix} R_4 - 6.28604 \\ 0 \\ 1.15470 + 1.92516 R_4 \end{bmatrix} \right),$$

from which the physical properties of $\wp_{13}$ can directly be obtained as

$$\boldsymbol{n}_{13} = 0.46095[\ 1\ \ 0\ \ 1.92516\ ]^T,$$

$$\zeta_{13}^{\perp} = \boldsymbol{N}_{13}\boldsymbol{m}_{13}^{o} = -[\ 0\ \ 2.81675\ \ 0\ ]^T,$$

$$\lambda_{13} = \boldsymbol{n}_{13}^T(\boldsymbol{v}_2 - \boldsymbol{v}_1) = -6.48792,$$

$$p_{13} = \boldsymbol{n}_{13}^T\boldsymbol{m}_{13}^{o} = (0.46095)^2(4.70624R_4 - 4.06305),$$

since

$$p_{13} = \frac{\lambda_{13}}{2\tan(\theta_{13}/2)} + R_4,$$

then we have

$$\theta_{13} = 2.62138 \text{ rad}.$$

This result shows clearly that the constant R_4 dose not enter into the picture of the physical properties of $\wp_{13}$. Note also that the same results can be obtained by using equation (19) with $\alpha = \alpha_3 = 2.06105$ rad.

11. Conclusions

In this paper, using only the ordinary linear algebra, a simplified form of the expression for a screw matrix, and the properties of the corresponding physical screw, are introduced. Using this transformation matrix, analytic description of the set of all available finite twists associated with the displacement of a line with a point from one spatial position to another is obtained. Based on the Sticher's selection of coordinate system, the finite screw cylindroid and many of its basic properties are re-derived. This selection makes it possible to derive a general expression for the pitches of the finite screw cylindroid to give an answer to the question as to whether there exist different appropriate measures of pitch other than Parkin's under which all the available finite twists from a 2-system. It is found that Parkin's pitch plus an arbitrary constant term is the only suitable measure of pitch. Further this constant term may be taken as zero without changing the physical properties of the twists in the cylindroid, or violating the existence of the 2-system. With this very definite result one can state that the expression of the pitch given by Parkin is not only sufficient for the finite screw cylindroid to be described as a 2-system but it is also a necessary condition. Furthermore, it is found that the determined 2-system can be considered as a linear combination of two special screws which are called in this paper the $\alpha = 0$ and the $\alpha = \pi$ screws. These two special screws, with the constant pitch term taken as zero, correspond, respectively, to Parkin's X and π key screws.

In conclusion, this paper has proved that Parkin's pitch can be considered as a necessary and sufficient measure for the pitches of the finite screw cylindroid under which the cylindroid form a 2-system of screws. Moreover, in the paper a compact analytical description of the proposed 2-system which resides on the finite screw cylindroid is introduced.

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