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Numerical Treatment of Strongly Elliptic Integral Equation

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ABSTRACT: The numerical treatment of boundary integral equations in the form of boundary element methods has became very popular and powerful tool for engineering computations of boundary value problems, in addition to finite difference and finite element methods. Here, we present some of the most important analytical and numerical aspects of the boundary integral equation. The concept of the principle symbol allows the characterization of the boundary integral equation whose variational formulation on the boundary provides there a Gãrding inequality. Therefore, the Galerkin method can be analyzed similarly to the domain finite element methods providing asymptotic convergence if the number of grid points increases. These asymptotic error analysis will be presented in details. To illustrate the efficiency of the Galerkin boundary element method we consider as an numerical experiment the strongly elliptic boundary integral equation with the logarithmic single layer potential. Consequently, we use the Gaussian elimination method as a direct solver and the conjugate gradient iteration to solve the positive definite linear system. A comparison is drawn between these methods.

KEYWORDS*:* Strong ellipticity, variational formulation, boundary element method, Galerkin scheme.

I. INTRODUCTION

Although the reduction of elliptic boundary value problems to equivalent integral equations on the boundary represents historically the earliest method of corresponding mathematical analysis, its numerical exploitation has been developed only more recently creating many activities in computational mathematics and engineering from several different fields to implement boundary integral methods. Therefore, the corresponding integral equations form now a much larger class than the classical Fredholm integral equations of the second kind with weakly singular kernels. They contain singular integral equations with Cauchy respectively Giraud kernels in elasticity and thermoelasticity, Fredholm integral equations of the first kind with weakly singular kernels as in elasticity, flow problems, electrostatics and conformal mapping and integro-differential operators with non-integrable kernels as in acoustics or elasticity. Whereas in classical analysis these types of equations have been treated differently, modern Fourier analysis of pseudo–differential operators allow us to formulate unifying properties which provide also an analysis of numerical methods for their approximate solution. The concept of the principle symbol allows the characterization of boundary integral equations whose variational formulation on the boundary provides there a Gãrding inequality. Therefore Galerkin boundary element method can be analyzed similarly to the domain finite element methods providing asymptotic convergence if the number of grid points increases. In engineering codes, however, mostly point collection is used for boundary element methods. Here stability is much more difficult to prove. The asymptotic error analysis based on the Galerkin formulation will be presented. The paper is organized as follows: In section 2 we present some important aspects of the variational formulation and strong ellipticity for the boundary integral equation. As in finite element methods for elliptic boundary value problems,

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also in boundary element methods, the variational formulation and coerciveness properties provide the basic mathematical foundations for rigorous error and convergence analysis. The variational formulation rests on the weak formulation of the boundary integral equation on Γ. Section 3 is devoted to the numerical handling of the boundary integral equation. Thus, we describe the boundary element method based on the Galerkin discretization of the boundary integral operator. Some asymptotic error results based on Galerkin formulation are presented. In section 4, we illustrate with an example the Galerkin boundary element method for constructing the solution of the single layer potential integral equation with Logarithmic kernel. Consequently, the conjugate gradient method (cg–method) is implemented to solve the system of algebraic equations. This turns out to be the most efficient method for solving symmetric and positive definite systems.

II. VARIATIONAL FORMULATION AND STRONG ELLIPTICITY

Consider the strongly elliptic boundary integral equation

$$A\,u = f \qquad \text{on} \quad \Gamma \tag{2.1}$$

where A is a $p \times p$ matrix of linear operator mapping the $p-\text{vector}$ valued function on Γ into $p-\text{vector}$ valued function, f is $p-\text{vector}$ valued function and u is the unknown function. Γ is a given compact sufficiently smooth manifold in R^2. The corresponding weak formulation of (2.1), reads as to find u such that for all test functions w there holds on Γ

$$\int_\Gamma (A\,u)\,\overline{w}\,d\Gamma = \int_\Gamma f\,\overline{w}\,d\Gamma \tag{2.2}$$

To analyze the bilinear form on the left-hand side of (2.2) and the mapping properties involved we introduce the Sobolev spaces of functions on Γ associated with the scalar products

$$(f,g)_{H^{\pm}(\Gamma)} = \sum_{k\in Z} \left| ¢_{ok} + k \right|^{2\pm} \hat{f}_k \quad for\ n = 2\,,\ \pm \in R \tag{2.3}$$

and

$$\langle f,g \rangle_{H^{\pm}(\Gamma)} = \sum_{|\ell| \le [\pm]} \sum_{j=1}^{M} \int_{\Gamma_j} \partial_t^\ell\, f\, \overline{\partial_t^\ell\, g}\, d\Gamma + \left(1 - ¢_{\pm[\pm]}\right) \sum_{|\ell| = [\pm]} \sum_{j=1}^{M} \int_{\Gamma_J} \int_{\Gamma_J} \frac{\left(\partial_t^\ell\, f(\vec{x}) - \partial_t^\ell\, f(\vec{y})\right)\left(\partial_t^\ell\, g(\vec{x}) - \partial_t^\ell\, g(\vec{y})\right)}{\left|\vec{x} - \vec{y}\right|^{n-1+2(\pm - [\pm])}} d\Gamma_x\, d\Gamma_y \tag{2.4}$$

for $n \ge 2$ and $\pm \ge 0$. Here ∂_t^ℓ denote all covariant derivatives of order $|\ell|$ in Γ and $\bigcup_{j=1}^{M} \Gamma_j$ is a finite covering of Γ by regular charts. $[\bullet]$ denotes the Gaussian bracket. Then the norms

$$\| f \|_{H^{\pm}(\Gamma)} = (f\,,\,f)_{H^{\pm}(\Gamma)}^{1/2} \tag{2.5}$$

and

$$\| f \|_{H^{\pm}(\Gamma)} = \langle f , f \rangle^{1/2}_{H^{\pm}(\Gamma)} \tag{2.6}$$

turn out to be equivalent. For $\pm < 0$ we use the duality with respect to $L^2(\Gamma)$,

$$\| f \|_{H^{\pm}(\Gamma)} = \sup_{\| \cdot \|_{H^{\pm}(\Gamma)} \leq 1} \left| \langle f , ¨ \rangle_{L^2(\Gamma)} \right| \tag{2.7}$$

where

$$\langle f , ¨ \rangle_{L^2(\Gamma)} = \int_{\Gamma} f \, \overline{¨} \, d\Gamma \tag{2.8}$$

Since the boundary integral operator considered here is also pseudo–differential operator, thus we have the following mapping property:

Theorem 2.1([11]) For a pseudo–differential operator A of order $\sim \in R$ and any $\pm \in R$ the mapping

$$A : H^{\pm}(\Gamma) \rightarrow H^{\pm - \sim}(\Gamma) \tag{2.9}$$

is continuous. For the proof see, for example, reference [11].

Since we are interested in solving (2.1), the continuity (2.9) is not enough. We also need some properties that provide the existence of A^{-1}. The following property is strong ellipticity defined as

Definition: A $p \times p$ matrix of pseudo–differential operator A of order $\sim$ is called strongly elliptic if there exist a positive constant k_0 and a complex–valued function $\cdot \in C^{\infty}(\Gamma)$ such that for all $\vec{x} \in \Gamma$ and all $¸ \in R^{n-1}$ with $|¸| = 1$ and all $s \in C^p$ the principle symbol of A satisfies

$$\mathrm{Re}\, s^T \big(\cdot (\vec{x}) \, a_0(\vec{x}, ¸) \big) \bar{s} \geq k_0 |s|^2 \tag{2.10}$$

Lemma 2.1: ([15,16]) For n = 2 the strong ellipticity (2.10) is equivalent to the property: For all $\vec{x} \in \Gamma$ and all $ª \in [0,1]$, the principle symbol a_0 satisfies

$$\mathrm{Det} \left\{ ª \, a_0(\vec{x}, 1) + (1 - ª) \, a_0(\vec{x}, -1) \right\} \neq 0 \tag{2.11}$$

For proof see, for example, [15,16]. The proposed coerciveness results read as follows:

Theorem 2.2: ([8]) If A is a strongly elliptic $p \times p$ system of pseudo–differential operator of order $\sim$ then there exist a positive constant k_1 and a completely continuous operator $T : H^{\sim/2}(\Gamma) \rightarrow H^{\sim/2}(\Gamma)$ such that for all $w \in H^{\sim/2}(\Gamma)$

$$\text{Re}\ \left\langle \left(\cdot\ \ A+T\right)w,w\right\rangle_{L^2(\Gamma)} \geq k_1 \left\| w \right\|^2_{H^{\cdot/2}(\Gamma)}\ . \tag{2.12}$$

In other words, $K = \cdot\ \ A+T$ is $H^{\sim/2}(\Gamma)-\text{elliptic}$ on Γ . The proof of (2.12) is based on (2.10) and can also be found in [5].

III. GALERKIN BOUNDARY ELEMENT METHOD

In order to solve the boundary integral equation (2.1) numerically we are going to use the boundary element method based on the Galerkin discretization scheme. Thus we start by introducing a family of finite- dimensional boundary element spaces H_h on Γ by choosing a regular $S_h^{d,r}$ family of boundary elements in the sense of Babuska and Aziz [6], $r < d$. More specifically, we use the global parametric representation

$$¶ = \cdot^{\rightarrow}(t) \quad , \quad \vec{x} = \cdot^{\rightarrow}(r) = \cdot^{\rightarrow}(r \mp 1) \tag{3.1}$$

of Γ and introduce 1–periodic piecewise $(d-1)$–degree polynomial splines on a family of partitions $\left\{0 = t_0 < t_1 < .... < t_M = 1\right\}$ of the unit interval, $h = \max\limits_{j=1,....,M}\left\{t_j - t_{j-1}\right\}$. Then, with the parametric representation (3.1), we transplant the splines onto Γ . Boundary integrals can then be evaluated in the local coordinates in which the finite elements appear as simple functions. The classical Galerkin method for solving (2.1) or (2.2) approximately uses H_h for the trial as well as for the test space, i.e. find $u_h \in H_h$ such that for all $w_h \in H_h$:

$$\langle A u_h, w_h \rangle_{L^2(\Gamma)} = \langle f, w_h \rangle_{L^2(\Gamma)} \tag{3.2}$$

Let $\left\{« _j\right\}_{j=1}^N$ denote a Basis of H_h . Then the desired approximation solution has the form

$$u_h(\vec{x}) = \sum_{j=1}^{N} ¥_j\ «_j(\vec{x}) \tag{3.3}$$

Where the coefficients $¥_j$ are to be determined by solving the quadratic finite system of linear equations

$$\sum_{j=1}^{N} \langle A «_j, «_k \rangle_{L^2(\Gamma)}\ ¥_j = \langle f, «_k \rangle_{L^2(\Gamma)}\ , k = 1,2,..., N \tag{3.4}$$

These equations are equivalent to (3.2). Since $A : H^{\sim/2}(\Gamma) \rightarrow H^{-\sim/2}(\Gamma)$, the bilinear form (3.2) and the influence matrix (3.4) will be well defined only if we require the conformity condition, $H_h \subset H^{\sim/2}(\Gamma)$. For the consistency of the Galerkin approximation (3.2) we also require the approximation property

$$\lim_{h\to 0}\ \inf_{u_h \in H_h} \left\| w - u_h \right\|_{H^{\sim/2}(\Gamma)} = 0 \qquad (3.5)$$

As is well known for linear problems, the convergence $u \to u_h$ can only be established if the approximate equations (3.4) are stable, which can be formulated in terms of the Ladyzenskaya–Babuska–Brezzi condition, in short LBB–condition : There exists ¥ > 0 such that for all $w_h \in H_h$ and the whole family H_h with $h \to 0$

$$\sup_{0 \neq w_h \in H_h} \left| \langle A v_h , w_h \rangle \right| \frac{1}{\left\| w_h \right\|_{H^{\sim/2}(\Gamma)}} \geq ¥ \left\| v_h \right\|_{H^{\sim/2}(\Gamma)} \qquad (3.6)$$

Lemma 3.1: (Cea's Lemma [7]) If the LBB–condition holds then Galerkin equations (3.2) and (3.4) are uniquely solvable and we have the quasi–optimal error estimate

$$\left\| u - u_h \right\|_{H^{\sim/2}(\Gamma)} \leq c \inf_{w_h \in H_h} \left\| u - w_h \right\|_{H^{\sim/2}(\Gamma)} \qquad (3.7)$$

Where the constant c is independent of u and h.

Proof: i) Since $A v_h = 0$ implies with (3.6) also $v_h = 0$, and since (3.4) is a system of quadratic linear equations, the uniqueness implies solvability.

ii) Due to the previous arguments, the solution u_h of (3.4) exists for $h > 0$ satisfying

$$\langle A u_h , w_h \rangle_{L^2(\Gamma)} = \langle A u , w_h \rangle_{L^2(\Gamma)} \quad \text{for all } w_h \in H_h \qquad (3.8)$$

Hence, the mapping $u \mapsto u_h := G_h u$, the Galerkin projection G_h exists for every h. Moreover, for every $u \in H^{\sim/2}(\Gamma)$, we find that G_h is a projection,

$$G_h v_h = v_h \text{ for } v_h \in H_h \text{ i.e. } G_h \big|_{H_h} = I \big|_{H_h} \qquad (3.9)$$

Moreover, for every $u \in H^{\sim/2}(\Gamma)$ we have with (3.6)

$$\left\| G_h u \right\|_{H^{\sim/2}(\Gamma)} = \left\| u_h \right\|_{H^{\sim/2}(\Gamma)} \leq \frac{1}{¥} \left| \langle A u_h , w_h^* \rangle \right|$$

with a specific $w_h^* \in H_h$, $\left\| w_h^* \right\|_{H^{\sim/2}(\Gamma)} = 1$ since on the finite–dimensional unit–sphere the supremum (3.6) becomes maximum. Inserting (3.8) into (3.9) and using continuity of $L^2 -$ duality and of A , we obtain

$$\|G_h\, u\|_{H^{\sim/2}(\Gamma)} \le \frac{1}{¥}\left|\langle A\,u_h, w_h^*\rangle\right| = \frac{1}{¥}\left|\langle A\,u_h, w_h^*\rangle\right|$$
$$\le c\,\|A\,u\|_{H^{-\sim/2}(\Gamma)}\,\|w_h^*\|_{H^{\sim/2}(\Gamma)} \le c'\|u\|_{H^{\sim/2}(\Gamma)} \qquad (3.10)$$

where the constant c' is independent of h and u. Hence, the LBB–condition (3.6) indeed implies stability. For (3.7) we use the inequality

$$\|u - u_h\|_{H^{\sim/2}(\Gamma)} = \|u - w_h + G_h\, w_h - G_h\, u\|_{H^{\sim/2}(\Gamma)}$$
$$\le (1 + c')\,\|u - w_h\|_{H^{\sim}(\Gamma)}$$

for every $w_h \in H_h$.

Theorem 3.1: Let $A: H^{\sim/2}(\Gamma) \to H^{-\sim/2}(\Gamma)$ be a bijective, strongly elliptic of pseudo–differential operator of order $\sim$ on Γ; let (3.5) and one of the following additional properties be satisfied:

i) $\cdot$ = constant and $T = 0$ in (2.12)

ii) $\cdot$ = constant ,

iii) The boundary elements provide in addition to (3.5) the property:

For every $¨ \in C^{\infty}(\Gamma)$ exist constants $¢ > 0$ and $C_{¨} \ge 0$ such that for all $w_h \in H_h$,

$$\|¨\, w_h - P_h\, ¨\, w_h\|_{H^{\sim/2}(\Gamma)} \le C_{¨}\, h^{¢}\,\|w_h\|_{H^{\sim/2}(\Gamma)} \qquad (3.11)$$

where $P_h: H^{\sim/2}(\Gamma) \to H_h$ denotes the $L^2(\Gamma)$ projection. Then there exists $h_0 > 0$ such that the boundary element Galerkin method (3.2) is stable satisfying the LBB–condition (3.6) for all $0 < h \le h_0$. Sometimes (3.11) is also called a super-approximation property. This property allows the application of Korn's trick, i.e. the freezing coefficient technique, also called the localization principle, to the boundary element methods; see references [1] and [2] for more details.

Proof: In order to show (3.6) we choose any v_h and then try to find some associated w_h for which the inequality (3.6) holds.

i) Here $K = \cdot\; A$ is $H^{\sim/2}(\Gamma)$–ellipti and we can choose $w_h = \cdot^{\,*}\, v_h \in H_h$ since here $\cdot$ is constant. Then, with $T = 0$ and (2.12) we obtain

$$\left|\langle A\,v_h, w_h\rangle\right| = \left|\langle A\,v_h, \cdot^{\,*} v_h\rangle\right| = \left|\langle \cdot\; A\,v_h, v_h\rangle\right|$$
$$\ge \operatorname{Re}\langle \cdot\; A\,v_h, v_h\rangle \ge ¥\,\|v_h\|^2_{H^{\sim/2}(\Gamma)}$$

With $\|v_h\|_{H^{\sim/2}(\Gamma)} = \|\cdot^{\,*-1}\, w_h\|_{H^{\sim/2}(\Gamma)} \le c_0\,\|v_h\|_{H^{\sim/2}(\Gamma)}$, $c_0 > 0$

The above inequality yields

$$\left|\langle A v_h, w_h \rangle\right| \geq \frac{\text{¥}}{c_0} \| v_h \|_{H^{\sim/2}(\Gamma)} \| w_h \|_{H^{\sim/2}(\Gamma)} ,$$

which implies (3.6).

ii) For this case, our proof follows the corresponding finite element version (see for example [12]). From (i) we already have for $K = \cdot\ A$ that the corresponding Galerkin projections $G_h w = \left(p_h K\ p_h \right)^{-1}\Big|_{H_h} p_h\ K\ w$ are uniformly bounded. Furthermore,

$$\lim_{h \to 0} \underset{\sim}{G}_h\ w = w \text{ in } H^{\sim/2}(\Gamma) \text{ for every } w \in H^{\sim/2}(\Gamma) \qquad (3.12)$$

due to (3.5). Now choose $w_h = \cdot\ ^* \left\{ I - G_h K^{-1} T \right\} v_h$. Since for $\| v_h \|_{H^{\sim/2}(\Gamma)} \leq 1$ the set $K^{-1} T\ v_h$ is compact in $H^{\sim/2}(\Gamma)$ due to the complete continuity of T, we find from (3.12) $\left\|(I - \underset{\sim}{G}_h) K^{-1} T\ v_h \right\|_{H^{\sim/2}} \leq \text{£}(h) \| v_h \|_{H^{\sim/2}}$ with $\text{£}(h) \to 0$ for $h \to 0$. Hence with $\cdot\ A = K - T$ and (2.12)

$$\begin{aligned} \left|\langle A v_h, w_h \rangle\right| &= \left|\langle\ A v_h, \cdot\ ^* \left\{ I - \underset{\sim}{G}_h\ K^{-1}\ T \right\} v_h\ \rangle\right| \\ &= \left|\langle \cdot\ \ A v_h, \left\{ K^{-1}(K - T) + (I - \underset{\sim}{G}_h)^{-1}\ K\ T \right\} v_h \rangle\right| \\ &\geq \left|\langle \cdot\ \ A v_h, K^{-1} \cdot\ \ A v_h \rangle\right| - c\, \text{£}(h) \| v_h \|^2_{H^{\sim/2}(\Gamma)} \\ &\geq \operatorname{Re} \langle K(K^{-1} \cdot\ \ A v_h), (K^{-1} \cdot\ \ A v_h) \rangle - c\, \text{£}(h) \| v_h \|^2_{H^{\sim/2}(\Gamma)} \\ &\geq \text{¥} \left\| K^{-1} \cdot\ \ A v_h \right\|^2_{H^{\sim/2}(\Gamma)} - c\, \text{£}(h) \| v_h \|^2_{H^{\sim/2}(\Gamma)} . \end{aligned}$$

Since we assume the existence of A^{-1} and $\|w_h\|_{H^{\sim/2}(\Gamma)} \leq c_0\ \|v_h\|_{H^{\sim/2}(\Gamma)}$ with $c_0 > 0$, the above estimate implies $\left|\langle A v_h, w_h \rangle\right| \geq \left(\text{¥}_0 - c\, \text{£}(h)\right) \| v_h \|_{H^{\sim/2}(\Gamma)} \| w_h \|_{H^{\sim/2}(\Gamma)}$ with $\text{¥}_0 > 0$. Hence, there exists $h_0 > 0$ such that $\text{¥}_0 - c\, \text{£}(h) \geq \frac{\text{¥}_0}{2}$ for all $0 < h < h_0$ and (3.6) follows.

iii) Again $K = \cdot\ A + T$ is $H^{\sim/2}(\Gamma)$ elliptic and the corresponding projection $\underset{\sim}{G}_h$ is uniformly bounded in $H^{\sim/2}(\Gamma)$. Here take $w_h = p_h \cdot\ ^* \left\{ I - \underset{\sim}{G}_h\ K^{-1}\ T \right\} v_h$. Then

$\| w_h \|_{H^{\sim/2}(\Gamma)} \leq c_0 \| v_h \|_{H^{\sim/2}(\Gamma)}$ with $c_0 > 0$, and

$$\left|\langle A v_h, w_h \rangle\right| \geq \left|\langle A v_h, \left\{ I - \underset{\sim}{G_h} K^{-1} T \right\} v_h \rangle\right|$$
$$- \left|\langle A v_h, (p_h - I) \cdot {}^{*} \left\{ I - \underset{\sim}{G_h} K^{-1} T \right\} v_h \rangle\right|$$
$$\geq \left|\langle \cdot \; A v_h, K^{-1} \cdot \; A v_h \rangle\right|$$
$$- \left|\langle \cdot \; A v_h, (I - \underset{\sim}{G_h}) K^{-1} T v_h \rangle\right| - c' h^{¢} \left\| v_h \right\|^2_{H^{~/2}(\Gamma)}$$

The first two terms can be estimated in the same manner as in case (ii), which yields

$\left|\langle A v_h, w_h \rangle\right| \geq \frac{1}{c_0}\left(¥_0 - c £(h) - c' h^{¢} \right) \left\| v_h \right\|_{H^{~/2}(\Gamma)} \left\| w_h \right\|_{H^{~/2}(\Gamma)}$, which implies (3.6) for all h sufficiently small.

Combining the above results, we conclude that the Galerkin method for injective, strongly elliptic system of pseudo–differential or boundary integral equations under the assumptions of theorem 3.1 is eatable and converges quasi–optimally for $h \to 0$.
This result can further be improved by using the finite element approximation properties of $S_h^{d,r}$ –systems, $r,d \in \mathrm{N}_0, r \leq d-1$.

Approximation property: Let $² \leq \quad \leq d$ and $² < r + \frac{1}{2}$ for $n=2$ or $² \leq r$ for $n=3$. Then there exists a constant c such that for any $v \in H \;\; (\Gamma)$ there exists a sequence $¡_h \in S_h^{d,r}$ providing

$$\left\| v - ¡_h \right\|_{H^{²}(\Gamma)} \leq c h^{\;\;-²} \left\| v \right\|_{H \;\;(\Gamma)}. \tag{3.13}$$

Theorem 3.2: ([15,16]). Let A be a bijective strongly elliptic pseudo–differential operator of order $\sim$. Let $\sim < 2r+1$ for $n=2$ or $\sim \leq 2r$ for $n=3$. Further, let $\sim - d \leq ² \leq \frac{\sim}{2} \leq \quad \leq d$. Then we have the asymptotic error estimate of optimal order

$$\left\| u_h - u \right\|_{H^{²}(\Gamma)} \leq c h^{\;\;-²} \left\| u \right\|_{H \;\;(\Gamma)} \tag{3.14}$$

Proof: (i) For the special case $² = \frac{\sim}{2}$, the proposed inequality (3.14) follows from Cea's Lemma, i.e. (3.7) together with (3.13).
(ii) For $² \leq \frac{\sim}{2}$ we perform the well–known Aubin–Nitsche duality arguments as in [5]. First we observe that for bijective pseudo–differential operator $A_{\sim}$ on Γ also its adjoint A^{*} with respect to the

L^2 – duality is a bijective pseudo–differential operator of the same order. We therefore know that $A^* : H^{\sim -\text{²}}(\Gamma) \to H^{-\text{²}}(\Gamma)$ is continuous and bijective, in particular, for

$$A^* v = \text{¨} \quad \text{we have} \quad \|v\|_{H^{\sim -\text{²}}(\Gamma)} \le c \, \|\text{¨}\|_{H^{-\text{²}}(\Gamma)} \,. \qquad (3.15)$$

With the Galerkin equations we have

$$\begin{aligned}
\left| \langle u - u_h , \text{¨} \rangle_{L^2(\Gamma)} \right| &= \left| \langle A(u - u_h), v \rangle_{L^2(\Gamma)} \right| \\
&= \inf_{\text{¡}_h \in H_h} \left| \langle A(u - u_h), v - \text{¡}_h \rangle_{L^2(\Gamma)} \right| \\
&\le c \, \| u - u_h \|_{H^{\sim/2}(\Gamma)} \inf_{\text{¡}_h \in H_h} \| v - \text{¡}_h \|_{H^{-\sim/2}(\Gamma)} \\
&\le c' \, h^{\quad -\sim/2} \, \| u \|_{H \quad (\Gamma)} \cdot h^{\sim/2-\text{²}} \, \| v \|_{H^{\sim -\text{²}}(\Gamma)} \\
&\le c'' \, h^{\quad -\text{²}} \, \| u \|_{H \quad (\Gamma)} \, \| \text{¨} \|_{H^{-\text{²}}(\Gamma)}
\end{aligned}$$

provided $\sim -\text{²} \le d$. Here, c'' is independent of h, u and ¨. Since $H^{\text{²}}$ and $H^{-\text{²}}$ are dual with respect to $\langle \cdot , \cdot \rangle_{L^2}$, we eventually find the proposed estimate:

$$\| u - u_h \|_{H^{\text{²}}(\Gamma)} = \sup_{\| \text{¨} \|_{H^{-\text{²}}(\Gamma)}} \left| \langle u - u_h , \text{¨} \rangle \right| \le c'' h^{\quad -\text{²}} \, \| u \|_{H \quad (\Gamma)} \,.$$

IV. NUMERICAL EXPERIMENT

In our test numerical experiment, we consider the boundary integral equation

$$\big(Au \big)(x) = f(x) \quad \textit{for} \quad x \in \Gamma \qquad (4.1)$$

with the logarithmic single layer potential

$$\big(Au \big)(x) = -\frac{1}{2\text{®}} \int_{\Gamma} Log \left| x - y \right| u(y) \, d\Gamma_y \qquad (4.2)$$

where $\Gamma = \partial\Omega \subset R^2$ is a curve with domain $\Omega < 1$. The integral operator A is L^2 – elliptic, furthermore, A is self–adjoint, positive definite operator and satisfies the Gãrding inequality on Γ

(see, for example [13,14] for more details). The classical Galerkin method for solving (4.1) uses V_h for trial as well as for the test space. Find $u_h \in L^2(\Gamma)$ such that for all $w_h \in L^2(\Gamma)$:

$$\langle A u_h , w_h \rangle_{L^2(\Gamma)} = \langle f , w \rangle_{L^2(\Gamma)} \tag{4.3}$$

Let $\{«_j\}_{j=1}^{N}$ denote a basis of V_h. Then the desired approximate solution has the form

$$u_h(x) = \sum_{j=1}^{N} ¥_j \, «_j(x) \tag{4.4}$$

Where the coefficients $¥_j$ are to be determined by solving the quadratic finite system of linear equations

$$\sum_{j=1}^{N} \langle A«_j , «_k \rangle_{L^2(\Gamma)} ¥_j = \langle f , «_k \rangle_{L^2(\Gamma)} \quad for \ \ k = 1,2,\ldots,N \tag{4.5}$$

In fact (4.5) can be written in the following short form

$$S\,¥ = f \tag{4.6}$$

where $S = (S_{k\,j})_{k,j=1,\ldots,N} = -\frac{1}{2®} \int_{\Gamma_j} \int_{\Gamma_k} Log\left| x - y \right| «_j(x) «_k(y) \, d\Gamma_y \, d\Gamma_x$ for the influence matrix, $¥ = (¥_j)_{j=1,..,N}$ and $f = (f_j)_{j=1,..,N} = \int_{\Gamma_J} f(x) «_j(x) \, d\Gamma_x$ for the right-hand side of the discretized equation. The computation of the influence matrix $S = (S_{k\,j})_{k,j=1,\ldots,N}$ has been performed numerically using Gauss quadrature. Since the operator A is self–adjoint and positive definite, it follows that the matrix S is symmetric and positive definite. Hence we can implement the solution methods for the discrete equation (4.6), namely: The Gaussian elimination method as a direct solver and the conjugate gradient iteration. For more details on these methods see for example [3,4,9]. The cg–iteration is given by the following algorithm [9,10].

1. Choose an initial vector $¥_n^{(0)}$ and compute $r_0 = S_n ¥_n^{(0)} - f_n \quad , set \quad p_0 = r_0$.
2. For $k \geq 0$ compute

$$\sim_k = \frac{\left(r_k^T , p_k\right)}{p_k^T \, S_n \, p_k}$$

$$¥_n^{(k+1)} = ¥_n^{(k)} + \sim_n \, p_k$$

$$r_{k+1} = S_n \, f_n^{(k+1)}$$

3. Stop the calculation if

$$\frac{\| r_{k+1} \|_2}{\| r_k \|_2} < £$$

4. Otherwise compute

$$_k = \frac{r_{k+1}^T \, S_n \, p_k}{p_k^T \, S_n \, p_k}$$

$$p_{k+1} = r_{k+1} + \ _k \, p_k$$

The convergence of the cg–iteration is based on the following theorem.
Theorem 4.1 ([9]), see chapter 5) For the positive definite matrix S_n the conjugate gradient method converges and fulfils the following error estimate

$$\| e^{(i)} \|_{S_n} \leq 2 \left(\frac{(k(S_n)-1)^{1/2}}{(k(S_n)+1)^{1/2}} \right) \| e^{(0)} \|_{S_n}$$

where $\| e^{(i)} \|_{S_n} = \| ¥_n^{(i)} - ¥_n \|_{S_n}$ and $\| e^{(0)} \|_{S_n} = \| ¥_n^{(0)} - ¥ \|_{S_n}$ with the equivalent norm

$$\| a \|_{S_n} = \sqrt{(S_n \, a, \, a)} = \| S_n^{1/2} \, a \|_2 \text{ and } k(S_n) = \mathrm{cond}(S_n) = \frac{a_{\max}}{a_{\min}} .$$

Table 4.1 shows the numerical results for the discrete equation (4.6) using the Gaussian elimination method and the conjugate gradient iteration. It contains both the number of iteration steps and the CPU–time in seconds required by the cg–iteration in addition to the CPU-time required by the Gaussian elimination. The number N denotes the dimension parameter of the solved problem.

N	Gauss elimination	Cg-iteration	
	CPU-time seconds	No. of iterations	CPU-time seconds
128	1.56	7	0.16
256	12.91	7	0.41
512	109.67	7	1.20
1024	902.52	7	3.36

Table 4.1

It is evident from the numerical results shown in table 4.1 that the cg–iteration is more efficient than the Gaussian elimination method for solving positive definite systems. It requires both less number of iterations and CPU-time in comparison to the Gaussian elimination. This demonstrates that one of the characteristic features of cg-iteration is its fast convergence.

V. CONCLUSIONS

This article shed some light on some of the most important analytical and numerical aspects of the strongly elliptic boundary integral equation. For a two-dimensional boundary value problems, the explicit Fourier analysis leads to a new class of quadrature based modified collocation methods-qualocation providing higher–orders of convergence than the Ritz-Galerkin or collocation methods. In order to perform any of the boundary element methods we still have to carry out numerical integration for computing the weights

$$< A« _j , « _k >_{L^2(\Gamma)} \text{ or } (A« _j)(\vec{x}_k)$$

Since, in general, this cannot be done explicitly, the boundary Γ as well as these weights will be approximated and a corresponding perturbation analysis is needed. Based on Strang's lemma for bilinear forms, such an analysis can be carried out. Still, numerical integration of the above coefficients is one of the major problems when writing boundary element codes. For two-dimensional problems, Galerkin collocation provides a fully discretized numerical scheme. Combining a Taylor expansion of the kernels of the boundary integral operators with clustering of integration domains, one can combine the integration of the boundary potentials with multigrid methods for boundary integral equations and develop very fast and efficient solution procedure. This analysis will be carried out in our forthcoming article.

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