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NUMERICAL METHODS FOR SOLVING FUZZY FREDHOLM INTEGRAL EQUATION OF THE SECOND KIND

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Abstract: In this article some numerical methods, namely: the Taylor expansion method and the Trapezoidal method, have been investigated and implemented to solve a fuzzy Fredholm integral equation of the second kind. Consequently, we convert a linear fuzzy Fredholm integral equation of the second kind into a linear system of integral equations of the second kind in crisp case. To demonstrate the credibility of these numerical schemes we consider a numerical test example. The numerical results show to be in a close agreement with the exact solution.

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1. Introduction

The fuzzy integral equations have attracted the attention of many scientists and researchers in recent years, due to their wide range of applications, such as Fuzzy control, Fuzzy finance, approximate reasoning and economic systems, etc.

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The concept of integration of fuzzy functions was first introduced by Dubois and Prade [5]. Then alternative approaches were later suggested by Goetschel and Voxman [8], Kaleva [13], Matloka [17], Nanda [19], and others. While Goetschel and Voxman [8], and later Matloka [17], preferred a Riemann integral type approach, Kaleva [13], choose to define the integral of fuzzy function using the Lebesgue type concept for integration. Park et al. [22] have considered the existence of solution of fuzzy integral equation in Banach space. Wu and Ma [4] investigated the Fuzzy Fredholm integral equation of the second kind, which is one of the first applications of fuzzy integration. Due to the complexity of solving Fuzzy Fredholm integral equations analytically, numerical methods have been proposed. For instance, Maleknejad [16] solved the first kind Fredholm integral equation by using the sinc function. Parandin and Araghi [21] established a method to approximate the solution using finite and divided differences methods.

Jafarzadeh [12], solved linear fuzzy Fredholm integral equation with Upper bound on error by Splinders Interpolation. Altaie [1] used the Bernstein piecewise polynomial. Parandin and Araghi [11] proposed the approximate solution by using an iterative interpolation. Lotfi and Mahdiani [15] used Fuzzy Galerkin method with error analysis. Attari and Yazdani [3] studied the application of Homotopy perturbation method. Mirzaee, Paripour and Yari [18] presented direct method using Triangular Functions.

Gohary and Gohary [10], found an approximate solution for a system of linear fuzzy Fredholm integral equation of the second kind with two variables which exploit hybrid Legendre and block-pulse functions, and Legendre wavelets. Ziari, Ezzati and Abbasbandy [23], used Fuzzy Haar Wavelet. Ghanbari, Toushmalni and Kamrani [9], presented a numerical method based on block-pulse functions (BPFs). Amawi [2], has investigated some analytical and numerical solutions for the fuzzy Fredholm integral equation of the second kind.

In this article some numerical methods, namely: the Taylor expansion method and the Trapezoidal method, have been investigated and implemented to solve fuzzy Fredholm integral equation of the second kind. Using the parametric form of fuzzy numbers, the fuzzy linear Fredholm integral equation of the second kind can be converted to a linear system of Fredholm integral equations of the second kind in the crisp case.

The paper is organized as follows: In Section 2, fuzzy Fredholm integral equation of the second kind is introduced. The Taylor expansion method used to approximate solution of fuzzy Fredholm integral equation of the second kind is addressed in Section 3. In Section 4 we present the Trapezoidal method as a uniformly convergent iterative procedure. The proposed numerical methods are implemented using a numerical example by applying the MAPLE software in Section 5, conclusions are given in Section 6.

2. Fuzzy Fredholm Integral Equation

A standard form of the Fredholm integral equation of the second kind is given by [11]

$$g(t) = f(t) + \lambda \int_a^b k(s,t)g(s)ds, \tag{2.1}$$

where λ is a positive parameter, $k[s,t]$ is a function called the kernel of the integral equation defined over the square $G : [a,b] * [a,b]$ and $f(t)$ is a given function of $t \in [a,b]$.

Now, if $f(t)$ is a crisp function then (2.1) possess crisp solution and the solution is fuzzy if $f(t)$ is a fuzzy function. We introduce parametric form of a fuzzy Fredholm integral equation of the second kind. Let $(\underline{f}(t,r), \overline{f}(t,r))$ and $(\underline{g}(t,r), \overline{g}(t,r)), 0 \leq r \leq 1$ and $t \in [a,b]$ are parametric forms of $f(t)$ and $g(t)$ respectively, then the parametric form of fuzzy Fredholm integral equation of the second kind is as follows:

$$\begin{aligned} \underline{g}(t,r) &= \underline{f}(t,r) + \lambda \textstyle\int_a^b \underline{U}(s,r)ds, \\ \overline{g}(t,r) &= \overline{f}(t,r) + \lambda \textstyle\int_a^b \overline{U}(s,r)ds, \end{aligned} \tag{2.2}$$

where

$$\underline{U}(t,r) = \begin{cases} k(s,t)\underline{g}(s,r), & k(s,t) \geq 0 \\ k(s,t)\overline{g}(s,r), & k(s,t) < 0 \end{cases} \tag{2.3}$$

and

$$\overline{U}(t,r) = \begin{cases} k(s,t)\overline{g}(s,r), & k(s,t) \geq 0 \\ k(s,t)\underline{g}(s,r), & k(s,t) < 0 \end{cases} \tag{2.4}$$

for each $0 \leq r \leq 1$ and $a \leq s,t \leq b$.

We can see that (2.2) is a crisp system of linear Fredholm integral equations for each $0 \leq r \leq 1$ and $a \leq t \leq b$.

Definition 2.1. ([14]) The fuzzy Fredholm integral equations system of the second kind is of the form:

$$g_i(t) = f_i(t) + \sum_{j=1}^{m} (\lambda_{ij} \int_a^b k_{ij}(s,t)g_i(s)ds) \quad , i = 1, ..., m, \tag{2.5}$$

where s, t, λ are real constants and $s, t \in [a, b]$,$\lambda_{ij} \neq 0$ for $i, j = 1, , m$.

In the system (2.5), $g(t) = [g_1(t), ..., g_m(t)]^T$ is unknown function. Moreover, the fuzzy function $f_i(t)$ and kernel $k_{ij}(s, t)$ are known and assumed to be sufficiently differentiable functions with respect to all their arguments on the interval $[a, b]$.

Now, let the parametric forms of $f_i(t)$ and $g_i(t)$ are $(\underline{f}_i(t, r), \overline{f}_i(t, r))$ and $(\underline{g}_i(t, r), \overline{g}_i(t, r))$, $0 \leq r \leq 1$, $t \in [a, b]$, respectively.

We write the parametric form of the given fuzzy Fredholm integral equations system as follows:

$$\begin{aligned} &\underline{g}_i(t, r) = \underline{f}_i(t, r) + \sum_{j=1}^{m} \left(\lambda_{ij} \int_a^b \underline{U}_{i,j}(s, r) ds \right), \\ &\overline{g}_i(t, r) = \overline{f}_i(t, r) + \sum_{j=1}^{m} \left(\lambda_{ij} \int_a^b \overline{U}_{i,j}(s, r) ds \right), \quad i = 1, ..., m, \end{aligned} \tag{2.6}$$

where

$$\underline{U}_{i,j}(s, r) = \begin{cases} k_{i,j}(s, t)\underline{g}_j(s, r), & k_{i,j}(s, t) \geq 0 \\ k_{i,j}(s, t)\overline{g}_j(s, r), & k_{i,j}(s, t) < 0 \end{cases} \tag{2.7}$$

and

$$\overline{U}_{i,j}(s, r) = \begin{cases} k_{i,j}(s, t)\overline{g}_j(s, r), & k_{i,j}(s, t) \geq 0 \\ k_{i,j}(s, t)\underline{g}_j(s, r), & k_{i,j}(s, t) < 0. \end{cases} \tag{2.8}$$

3. Taylor Expansion Method

This method is based on differentiating p-times both sides of the linear fuzzy Fredholm integral equation of the second kind and then substitute the Taylor series expansion for the unknown function into the integral equation. As a result, we obtain a linear system for which the solution of this system yields the unknown Taylor coefficients of the solution functions.

If we assume that [11]

$$\begin{cases} \lambda_{ij} k_{i,j}(s, t) \geq 0, & a \leq s \leq c_{i,j}, \\ \lambda_{ij} k_{i,j}(s, t) < 0, & c_{i,j} \leq s \leq b, \end{cases}$$

then system (2.6) can be transformed into

$$\begin{aligned}\underline{g}_i(t,r) &= \underline{f}_i(t,r)\\ &+\sum_{j=1}^{m}\lambda_{ij}\left(\int_a^c k_{i,j}(s,t)\underline{g}_j(s,r)ds+\int_c^b k_{i,j}(s,t)\overline{g}_j(s,r)ds\right),\\ \overline{g}_i(t,r) &= \overline{f}_i(t,r)\\ &+\sum_{j=1}^{m}\lambda_{ij}\left(\int_a^c k_{i,j}(s,t)\overline{g}_j(s,r)ds+\int_c^b k_{i,j}(s,t)\underline{g}_j(s,r)ds\right),\\ &\qquad i=1,...,m.\end{aligned}\tag{3.1}$$

Differentiating both sides of each equation of the system (3.1) N-times with respect to t, we get

$$\begin{aligned}&\frac{\partial^{(p)}\underline{g}_i(t,r)}{\partial t^p}=\frac{\partial^{(p)}\underline{f}_i(t,r)}{\partial t^p}\\ &+\sum_{j=1}^{m}\lambda_{i,j}\left(\int_a^c\frac{\partial^{(p)}k_{i,j}(s,t)}{\partial t^p}\underline{g}_j(s,r)ds+\int_c^b\frac{\partial^{(p)}k_{i,j}(s,t)}{\partial t^p}\overline{g}_j(s,r)ds\right),\\ &\frac{\partial^{(p)}\overline{g}_i(t,r)}{\partial t^p}=\frac{\partial^{(p)}\overline{f}_i(t,r)}{\partial t^p}\\ &+\sum_{j=1}^{m}\lambda_{i,j}\left(\int_a^c\frac{\partial^{(p)}k_{i,j}(s,t)}{\partial t^p}\overline{g}_j(s,r)ds+\int_c^b\frac{\partial^{(p)}k_{i,j}(s,t)}{\partial t^p}\underline{g}_j(s,r)ds\right),\\ &\qquad i,j=1,...,m,\\ &\qquad p=0,...,N.\end{aligned}\tag{3.2}$$

Let us introduce the following notations for shortness:

$$\begin{aligned}\underline{g}_i^{(p)}(z,r)&=\tfrac{\partial^{(p)}\underline{g}_i(t,r)}{\partial t^p}|_{t=z}, & \overline{g}_i^{(p)}(z,r)&=\tfrac{\partial^{(p)}\overline{g}_i(t,r)}{\partial t^p}|_{t=z},\\ \underline{f}_i^{(p)}(z,r)&=\tfrac{\partial^{(p)}\underline{f}_i(t,r)}{\partial t^p}|_{t=z}, & \overline{f}_i^{(p)}(z,r)&=\tfrac{\partial^{(p)}\overline{f}_i(t,r)}{\partial t^p}|_{t=z},\\ k_{i,j}^{(p)}(z,r)&=\tfrac{\partial^{(p)}k_{i,j}(t,r)}{\partial t^p}|_{t=z}, & &i,j=1,...,m.\end{aligned}\tag{3.3}$$

Now, we expand the unknown functions $\underline{g}_i(s,r)$ and $\overline{g}_i(s,r)$ in Taylor series for multivariate variable at arbitrary point z, neglecting the truncation error, and get

$$\begin{aligned}\underline{g}_{j,N}&=\textstyle\sum_{q=0}^{N}\frac{1}{q!}\underline{g}_i^{(p)}(s,r)(s-z)^p,\\ \overline{g}_{j,N}&=\textstyle\sum_{q=0}^{N}\frac{1}{q!}\overline{g}_i^{(p)}(s,r)(s-z)^p,\quad a\le z\le b, j=1,...,m.\end{aligned}\tag{3.4}$$

Substituting equations (3.4) and (3.3) into equation (3.2) yields:

$$\underline{g}_i^{(p)}(z,r) = \underline{f}_i^{(p)}(z,r) + \sum_{j=1}^{m}\left(\sum_{q=0}^{N}\frac{\lambda_{ij}}{q!}\int_a^c k_{i,j}^{(p)}(s,z)\underline{g}_j(z,r)(s-z)^p ds \right.$$
$$\left. + \sum_{q=0}^{N}\frac{\lambda_{ij}}{q!}\int_c^b k_{i,j}^{(p)}(s,z)\overline{g}_j(z,r)(s-z)^p ds\right),$$

$$\overline{g}_i^{(p)}(z,r) = \overline{f}_i^{(p)}(z,r) + \sum_{j=1}^{m}\left(\sum_{q=0}^{N}\frac{\lambda_{ij}}{q!}\int_a^c k_{i,j}^{(p)}(s,z)\overline{g}_j(z,r)(s-z)^p ds \right.$$
$$\left. + \sum_{q=0}^{N}\frac{\lambda_{ij}}{q!}\int_c^b k_{i,j}^{(p)}(s,z)\underline{g}_j(z,r)(s-z)^p ds\right). \quad (3.5)$$

By using the notations:

$$\begin{aligned} w_{p,q}^{(i,j)} &= \tfrac{\lambda_{ij}}{q!}\int_a^{c_{i,j}} k_{i,j}^{(p)}(s,z)(s-z)^p ds, \quad i,j=1,...,m, \\ w_{p,q}{'}^{(i,j)} &= \tfrac{\lambda_{ij}}{q!}\int_{c_{i,j}}^{b} k_{i,j}^{(p)}(s,z)(s-z)^p ds, \quad p,q=0,...,N, \end{aligned} \quad (3.6)$$

equation (3.5) can be written in the form

$$\begin{aligned} \underline{g}_i^{(p)}(z,r) =& \underline{f}_i^{(p)}(z,r) \\ &+ \sum_{j=1}^{m}\left(\sum_{q=0}^{N} w_{p,q}^{(i,j)}\underline{g}_j(z,r) + \sum_{q=0}^{N} w_{p,q}{'}^{(i,j)}\overline{g}_j(z,r)\right), \\ \overline{g}_i^{(p)}(z,r) =& \overline{f}_i^{(p)}(z,r) \\ &+ \sum_{j=1}^{m}\left(\sum_{q=0}^{N} w_{p,q}^{(i,j)}\overline{g}_j(z,r) + \sum_{q=0}^{N} w_{p,q}{'}^{(i,j)}\underline{g}_j(z,r)\right). \end{aligned} \quad (3.7)$$

We can rearrange (3.7) as follows:

$$\begin{aligned} -\underline{f}_i^{(p)}(z,r) =& -\underline{g}_i^{(p)}(z,r) \\ &+ \sum_{j=1}^{m}\left(\sum_{q=0}^{N} w_{p,q}^{(i,j)}\underline{g}_j(z,r) + \sum_{q=0}^{N} w_{p,q}{'}^{(i,j)}\overline{g}_j(z,r)\right), \\ -\overline{f}_i^{(p)}(z,r) =& -\overline{g}_i^{(p)}(z,r) \\ &+ \sum_{j=1}^{m}\left(\sum_{q=0}^{N} w_{p,q}^{(i,j)}\overline{g}_j(z,r) + \sum_{q=0}^{N} w_{p,q}{'}^{(i,j)}\underline{g}_j(z,r)\right). \end{aligned} \quad (3.8)$$

When $p = q$, (3.8) becomes:

$$\begin{aligned} -\underline{f}_i^{(p)}(z,r) &= \sum_{j=1}^{m}\sum_{q=0}^{N}\left(w_{p,q}{}^{(i,j)}-1\right)\underline{g}_j(z,r), \\ -\overline{f}_i^{(p)}(z,r) &= \sum_{j=1}^{m}\sum_{q=0}^{N}\left(w_{p,q}{}^{(i,j)}-1\right)\overline{g}_j(z,r). \end{aligned} \tag{3.9}$$

Consequently, equation (3.7) can be written in the following matrix form:

$$WG = F, \tag{3.10}$$

where

$$G = \begin{bmatrix} \underline{g}_1(z,r) \\ \vdots \\ \underline{g}_i^{(N)}(z,r) \\ \overline{g}_1(z,r) \\ \vdots \\ \overline{g}_i^{(N)}(z,r) \\ \vdots \\ \underline{g}_m(z,r) \\ \vdots \\ \underline{g}_m^{(N)}(z,r) \\ \overline{g}_m(z,r) \\ \vdots \\ \overline{g}_m^{(N)}(z,r) \end{bmatrix}, F = \begin{bmatrix} \underline{f}_1(z,r) \\ \vdots \\ \underline{f}_i^{(N)}(z,r) \\ \overline{f}_1(z,r) \\ \vdots \\ \overline{f}_i^{(N)}(z,r) \\ \vdots \\ \underline{f}_m(z,r) \\ \vdots \\ \underline{f}_m^{(N)}(z,r) \\ \overline{f}_m(z,r) \\ \vdots \\ \overline{f}_m^{(N)}(z,r) \end{bmatrix}, W = \begin{bmatrix} W^{(1,1)} & \cdots & W^{(1,m)} \\ \vdots & \ddots & \vdots \\ W^{(m,1)} & \cdots & W^{(m,m)} \end{bmatrix}.$$

The parochial matrices $W^{(i,j)}$ are defined by the following elements (see [11])

$$W^{(i,j)} = \begin{bmatrix} W_{1,1}{}^{(i,j)} & W_{1,2}{}^{(i,j)} \\ W_{2,1}{}^{(i,j)} & W_{2,2}{}^{(i,j)} \end{bmatrix},$$

where

$$W_{1,1}{}^{(i,j)} = W_{2,2}{}^{(i,j)}$$

$$= \begin{bmatrix} W_{0,0}{}^{(i,j)}-1 & W_{0,1}{}^{(i,j)} & \cdots & W_{0,N-1}{}^{(i,j)} & W_{0,N}{}^{(i,j)} \\ W_{1,0}{}^{(i,j)} & W_{1,1}{}^{(i,j)}-1 & \cdots & W_{1,N-1}{}^{(i,j)} & W_{1,N}{}^{(i,j)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ W_{N-1,0}{}^{(i,j)} & W_{N-1,1}{}^{(i,j)} & \cdots & W_{N-1,N-1}{}^{(i,j)}-1 & W_{N-1,N}{}^{(i,j)} \\ W_{N,0}{}^{(i,j)}-1 & W_{N,1}{}^{(i,j)} & \cdots & W_{N,N-1}{}^{(i,j)} & W_{N,N}{}^{(i,j)}-1 \end{bmatrix},$$

$$W_{1,2}'^{(i,j)} = W_{2,1}'^{(i,j)} = \begin{bmatrix} W_{0,0}'^{(i,j)} & W_{0,1}'^{(i,j)} & \cdots & W_{0,N-1}'^{(i,j)} & W_{0,N}'^{(i,j)} \\ W_{1,0}'^{(i,j)} & W_{1,1}'^{(i,j)} & \cdots & W_{1,N-1}'^{(i,j)} & W_{1,N}'^{(i,j)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ W_{N-1,0}'^{(i,j)} & W_{N-1,1}'^{(i,j)} & \cdots & W_{N-1,N-1}'^{(i,j)} & W_{N-1,N}'^{(i,j)} \\ W_{N,0}'^{(i,j)} & W_{N,1}'^{(i,j)} & \cdots & W_{N,N-1}'^{(i,j)} & W_{N,N}'^{(i,j)} \end{bmatrix}.$$

The solution of system (3.10) is given as:

$$\begin{aligned} \underline{g}(t,r) &= \sum_{p=0}^{N} \frac{1}{p!} \frac{\partial^{(p)} \underline{g}(t,r)}{\partial t^p}\Big|_{t=z} (t-z)^p, \\ \overline{g}(t,r) &= \sum_{p=0}^{N} \frac{1}{p!} \frac{\partial^{(p)} \overline{g}(t,r)}{\partial t^p}\Big|_{t=z} (t-z)^p, \quad a \le z \le b. \end{aligned} \tag{3.11}$$

3.1. Convergence Analysis

In order to show the efficiency of the Taylor expansion method, one can show that the approximate solution converges to the exact solution of system (2.6) (see [11] for more details).

Theorem 3.1. ([11]) *If $\underline{g}_{j,N}(t,r)$ and $\overline{g}_{j,N}(t,r)$ are Taylor polynomials of degree N and their coefficients have been found by solving the linear system (3.10), then they converge to the exact solution of system (2.6), when $N \to \infty$.*

4. Trapezoidal Method

We compute the integral of a fuzzy function $g(t,r)$ using the Riemann integral of $\underline{g}(t,r)$ and $\overline{g}(t,r)$ by applying the trapezoidal rule. We consider $\underline{g}(t,r)$ and $\overline{g}(t,r)$ over the interval $[a,b]$, then subdivide the interval $[a,b]$ into n subintervals of equal width $h = \frac{(b-a)}{n}$ using equally spaced nodes:

$$\begin{aligned} a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b, \\ t_i = a + ih, t_i - t_{i-1} = h, 1 \le i \le n. \end{aligned} \tag{4.1}$$

We define (see [6])

$$\begin{aligned} \underline{S}_n(r) &= h\left[\frac{\underline{f}(a,r)+\underline{f}(b,r)}{2} + \sum_{i=1}^{n-1} \underline{f}(t_i,r)\right], \\ \overline{S}_n(r) &= h\left[\frac{\overline{f}(a,r)+\overline{f}(b,r)}{2} + \sum_{i=1}^{n-1} \overline{f}(t_i,r)\right]. \end{aligned} \tag{4.2}$$

Then, for arbitrary fixed r, we have

$$\begin{array}{l}\lim\limits_{n\longrightarrow\infty}\underline{S}_n(r)=\underline{g}(r)=\int_a^b\underline{f}(t,r)dt,\\ \lim\limits_{n\longrightarrow\infty}\overline{S}_n(r)=\overline{g}(r)=\int_a^b\overline{f}(t,r)dt.\end{array}\tag{4.3}$$

Theorem 4.1. ([7]) *If $f(t)$ is continuous in the metric D, then $\underline{S}_n(r)$ and $\overline{S}_n(r)$ converge uniformly in r to $\underline{g}(t,r)$ and $\overline{g}(t,r)$ respectively.*

Definition 4.1. ([6]) A fuzzy number $u(r)=(\underline{u}(r),\overline{u}(r))$ belongs to CE is defined as

$$CE=\{(\underline{u}(r),\overline{u}(r)):\underline{u}(r),\overline{u}(r)\in C[0,1]\},$$

where CE is a subclass of E.

Theorem 4.2. ([7]) *Let $g(t,r)=(\underline{g}(t,r),\overline{g}(t,r))$ be a fuzzy continuous function in t for fixed r and belong to CE, then its approximate solutions $\underline{S}_n(r)$ and $\overline{S}_n(r)$ converge uniformly.*

The exact iterative process for finding the exact solution for equation (2.1) is given by

$$\begin{array}{l}g_0(t)=f(t),\\ g_m(t)=f(t)+\lambda\int_a^b k(s,t)g_{m-1}(s)ds,\quad m\geq 1.\end{array}\tag{4.4}$$

However, the numerical process provide us with approximate fuzzy function for $g_m(t)$. If we denote it ${S_n}^{(m)}$ at the m-th iteration using n integration nodes, then we have

$${S_n}^{(m)}(t,r)=f(t,r)+\lambda\int_a^b k(s,t){S_{n-1}}^{(m-1)}ds+\delta_n(t,r),\tag{4.5}$$

where $\delta_n(t,r)=({\delta_n}^{(1)}(t,r),{\delta_n}^{(2)}(t,r))$ are uniformly convergent to 0 as $n,m\rightarrow\infty$.

Now, let $\delta_n(t,r)=(\underline{\delta}_n(t,r),\overline{\delta}_n(t,r))$ and neglect $\delta_n(t,r)$ in equation (4.5), we obtain

$$\begin{array}{l}{\underline{S}_n}^{(0)}(t,r)=\underline{f}(t,r),\\ {\underline{S}_n}^{(m)}(t,r)=\underline{f}(t,r)+\lambda\int_a^b k(s,t){\underline{S}_{n-1}}^{(m-1)}ds\end{array}\tag{4.6}$$

and

$$\begin{aligned}{\overline{S}_n}^{(0)}(t,r)&=\overline{f}(t,r),\\ {\overline{S}_n}^{(m)}(t,r)&=\overline{f}(t,r)+\lambda\int_a^b k(s,t){\overline{S}_{n-1}}^{(m-1)}ds.\end{aligned}\tag{4.7}$$

Theorem 4.3. ([7]) *Let $S_n{}^{(m)}(t)$ be an approximation to $g_m(t)$ using the trapezoidal rule with m equally spaced integration nodes, then $S_n{}^{(m)}(t)$ converges uniformly to the unique solution $g(t)$ when $n, m \to \infty$.*

5. Numerical Examples and Results

In order to test the efficiency of our numerical schemes, we have carried out some numerical experiments. We compare numerical results with exact solutions using the metric of Definition 5.1 (see Tables 1 and 2). Further comparison between the approximate solutions and the exact solutions can be seen in Figures 1 and 2 for fixed $t = 1$.

Definition 5.1. ([23]) For arbitrary fuzzy numbers $u, v \in E$ the quantity

$$D(u,v) = \sup_{0 \le r \le 1} \{\max |\underline{u}(r) - \underline{v}(r)|, \max |\overline{u}(r) - \overline{v}(r)|\}$$

defines the distance between u and v.

Example 1. (Taylor expansion method) The following fuzzy Fredholm integral equations:

$$\begin{aligned} \underline{g}(t,r) &= (r+1).(\exp(-t) + t - \sin t) + \int_0^1 \tfrac{1}{2}.\sin t.\underline{g}(s,s)ds, \\ \overline{g}(t,r) &= (3-r).(\exp(-t) + t - \sin t) + \int_0^1 \tfrac{1}{2}.\sin t.\overline{g}(s,s)ds \end{aligned} \tag{5.1}$$

have the exact solutions

$$\begin{aligned} \underline{g}(t,r) &= (r+1).(\exp(-t) + t), \\ \overline{g}(t,r) &= (3-r).(\exp(-t) + t). \end{aligned} \tag{5.2}$$

Here we expand the unknown functions $\underline{g}(t,r)$ and $\overline{g}(t,r)$ in Taylor series at $z = \frac{1}{2}$.

Algorithm (5.1):

1. input $a, b, \lambda_{i,j}, z, m, k_{i,j}(s,t), \underline{f}_i(t,r), \overline{f}_i(t,r)$
2. input the Taylor expansion degree N
3. calculate $\frac{\partial^{(p)} k_{i,j}(s,t)}{\partial t^p}, \frac{\partial^{(p)} \underline{f}_i(t,r)}{\partial t^p}, \frac{\partial^{(p)} \overline{f}_i(t,r)}{\partial t^p}, p, q = 0, ..., N$
4. calculate $w_{p,q}{}^{(i,j)} = \frac{\lambda_{ij}}{q!} \int_a^{c_{i,j}} k_{i,j}{}^{(p)}(s,z)(s-z)^p ds, i, j = 1, ..., m$

5. calculate ${w_{p,q}}'^{(i,j)} = \frac{\lambda_{ij}}{q!}\int_{c_{i,j}}^{b} {k_{i,j}}^{(p)}(s,z)(s-z)^p ds, p,q = 0,...,N$

6. put ${W_{1,1}}^{(i,j)} = {W_{2,2}}^{(i,j)}$

$$= \begin{bmatrix} {W_{0,0}}^{(i,j)}-1 & {W_{0,1}}^{(i,j)} & \cdots & {W_{0,N-1}}^{(i,j)} & {W_{0,N}}^{(i,j)} \\ {W_{1,0}}^{(i,j)} & {W_{1,1}}^{(i,j)}-1 & \cdots & {W_{1,N-1}}^{(i,j)} & {W_{1,N}}^{(i,j)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ {W_{N-1,0}}^{(i,j)} & {W_{N-1,1}}^{(i,j)} & \cdots & {W_{N-1,N-1}}^{(i,j)}-1 & {W_{N-1,N}}^{(i,j)} \\ {W_{N,0}}^{(i,j)}-1 & {W_{N,1}}^{(i,j)} & \cdots & {W_{N,N-1}}^{(i,j)} & {W_{N,N}}^{(i,j)}-1 \end{bmatrix}$$

7. put ${W_{1,2}}'^{(i,j)} = {W_{2,1}}'^{(i,j)}$

$$= \begin{bmatrix} {W_{0,0}}'^{(i,j)} & {W_{0,1}}'^{(i,j)} & \cdots & {W_{0,N-1}}'^{(i,j)} & {W_{0,N}}'^{(i,j)} \\ {W_{1,0}}'^{(i,j)} & {W_{1,1}}'^{(i,j)} & \cdots & {W_{1,N-1}}'^{(i,j)} & {W_{1,N}}'^{(i,j)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ {W_{N-1,0}}'^{(i,j)} & {W_{N-1,1}}'^{(i,j)} & \cdots & {W_{N-1,N-1}}'^{(i,j)} & {W_{N-1,N}}'^{(i,j)} \\ {W_{N,0}}'^{(i,j)} & {W_{N,1}}'^{(i,j)} & \cdots & {W_{N,N-1}}'^{(i,j)} & {W_{N,N}}'^{(i,j)} \end{bmatrix}$$

8. denote

$$W^{(i,j)} = \begin{bmatrix} {W_{1,1}}^{(i,j)} & {W_{1,2}}^{(i,j)} \\ {W_{2,1}}^{(i,j)} & {W_{2,2}}^{(i,j)} \end{bmatrix}$$

9. put $F = \begin{bmatrix} \underline{f}_1(z,r) \\ \vdots \\ {\underline{f}_i}^{(N)}(z,r) \\ \overline{f}_1(z,r) \\ \vdots \\ {\overline{f}_i}^{(N)}(z,r) \\ \vdots \\ \underline{f}_m(z,r) \\ \vdots \\ {\underline{f}_m}^{(N)}(z,r) \\ \overline{f}_m(z,r) \\ \vdots \\ {\overline{f}_m}^{(N)}(z,r) \end{bmatrix}, W = \begin{bmatrix} W^{(1,1)} & \cdots & W^{(1,m)} \\ \vdots & \ddots & \vdots \\ W^{(m,1)} & \cdots & W^{(m,m)} \end{bmatrix}$

10. solve the following linear system $WG = F$

11. Estimate $\underline{g}(z,r), \overline{g}(z,r)$ by computing Taylor expansion for G

$\underline{g}(t,r) = \sum_{p=0}^{N} \frac{1}{p!}\frac{\partial^{(p)}\underline{g}(t,r)}{\partial t^p}|_{t=z}(t-z)^p,$

$\overline{g}(t,r) = \sum_{p=0}^{N} \frac{1}{p!}\frac{\partial^{(p)}\overline{g}(t,r)}{\partial t^p}|_{t=z}(t-z)^p, \quad a \le z \le b, j = 1,...,m.$

Thus we obtain the following results:

$$W_{1,1}{}^{(i,j)} = W_{2,2}{}^{(i,j)}$$
$$= \begin{bmatrix} [-0.588105905 & 0.033765722 & 0.017721040 & 0.000848119 \\ 0.753967085 & -0.938192261 & 0.032438147 & 0.001552471 \\ -0.411894096 & -0.033765722 & -1.017721040 & -0.000848119 \\ -0.753967085 & -0.061807739 & -0.032438147 & -1.001552471] \end{bmatrix}$$

$$W_{1,1}{}^{(i,j)} = W_{2,2}{}^{(i,j)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

hence

$$W = \begin{bmatrix} W_{1,1}{}^{1,1} & W_{1,2}{}^{1,1} \\ W_{2,1}{}^{1,1} & W_{2,2}{}^{1,1} \end{bmatrix}$$

$$F = \begin{bmatrix} -\underline{f}(t,r) \\ -\underline{f}'(t,r) \\ -\underline{f}''(t,r) \\ -\underline{f}'''(t,r) \\ -\overline{f}(t,r) \\ -\overline{f}'(t,r) \\ -\overline{f}''(t,r) \\ -\overline{f}'''(t,r) \end{bmatrix} = \begin{bmatrix} -(r+1)(\exp(-t)+t-\sin t) \\ -(r+1)(-\exp(-t)+1-\cos t) \\ -(r+1)(\exp(-t)+\sin t) \\ -(r+1)(-\exp(-t)+\cos t) \\ -(3-r)(\exp(-t)+t-\sin t) \\ -(3-r)(-\exp(-t)+1-\cos t) \\ -(3-r)(\exp(-t)+\sin t) \\ -(3-r)(-\exp(-t)+\cos t) \end{bmatrix}_{t=\frac{1}{2}}$$

$$F = \begin{bmatrix} -0.627105121(r+1) \\ 0.484113225(r+1) \\ -1.085956198(r+1) \\ -0.271051902(r+1) \\ -0.627105121(3-r) \\ 0.484113222(3-r) \\ -1.085956198(3-r) \\ -0.271051902(3-r) \end{bmatrix}.$$

Solving the following linear system

$$WG = F,$$

we obtain

$$G = \begin{bmatrix} -\underline{g}(0.5,r) \\ -\underline{g}^{'}(0.5,r) \\ -\underline{g}^{''}(0.5,r) \\ -\underline{g}^{'''}(0.5,r) \\ -\overline{g}(0.5,r) \\ -\overline{g}^{'}(0.5,r) \\ -\overline{g}^{''}(0.5,r) \\ -\overline{g}^{'''}(0.5,r) \end{bmatrix} = \begin{bmatrix} 1.106288029(r+1) \\ 0.393025207(r+1) \\ 0.606773291(r+1) \\ 0.606086526(r+1) \\ 1.106288029(3-r) \\ 0.393025207(3-r) \\ 0.606773291(3-r) \\ -0.606086526(3-r) \end{bmatrix}.$$

Now,

$$\begin{aligned} \underline{g}(t,r) &= \sum_{p=0}^{N} \frac{1}{p!} \frac{\partial^{(p)} \underline{g}(t,r)}{\partial t^p}|_{t=\frac{1}{2}} (t-\frac{1}{2})^p \\ &= 0.998248889(r+1) + 0.013877746(r+1).t + \\ &\quad 0.454908277(r+1).t^2 - 0.101014421(r+1).t^3, \\ \overline{g}(t,r) &= \sum_{p=0}^{N} \frac{1}{p!} \frac{\partial^{(p)} \overline{g}(t,r)}{\partial t^p}|_{t=\frac{1}{2}} (t-\frac{1}{2})^p \\ &= 0.998248889(3-r) + 0.013877746(3-r).t + \\ &\quad 0.454908277(3-r).t^2 - 0.101014421(3-r).t^3. \end{aligned} \tag{5.3}$$

Figure 1 compares the exact solution and the approximate solution for a fixed $t = 1$.

r	$\underline{g}_{exact}$	$\overline{g}_{exact}$	$\underline{g}_{approximate}$	$\overline{g}_{approximate}$	$error = D(g_{exact}, g_{approximate})$
0.0	1.367879441	4.103638324	1.366020491	4.098061473	$5.576851372 \times 10^{-3}$
0.1	1.504667385	3.966850379	1.502622540	3.961459423	$5.390955481 \times 10^{-3}$
0.2	1.641455329	3.830062435	1.639224589	3.824857374	$5.205060586 \times 10^{-3}$
0.3	1.778243274	3.693274491	1.775826638	3.688255325	$5.019165700 \times 10^{-3}$
0.4	1.915031218	3.556486547	1.912428687	3.551653276	$4.833270801 \times 10^{-3}$
0.5	2.051819162	3.419698603	2.049030737	3.415051227	$4.647375913 \times 10^{-3}$
0.6	2.188607106	3.282910659	2.185632786	3.278449178	$4.461481016 \times 10^{-3}$
0.7	2.325395050	3.146122715	2.322234835	3.141847129	$4.275586125 \times 10^{-3}$
0.8	3.009334771	3.009334771	2.458836884	3.005245080	$4.089691232 \times 10^{-3}$
0.9	2.598970938	2.872546826	2.595438933	2.868643031	$3.903795339 \times 10^{-3}$
1.0	2.735758882	2.735758882	2.732040982	2.732040982	$3.717900446 \times 10^{-3}$

Table 1: The error resulted by algorithm (5.1) at $t = 1$

Example 2. (Trapezoidal method) The fuzzy Fredholm integral equations (5.1) have the exact solution (5.2) where $n = 51$, on the interval $[0,1], h = \frac{(b-a)}{n} = 0.01960784314, 0 = t_0 \leq t_1 \leq \cdots \leq t_{51} = 1, t_i = i.h.$

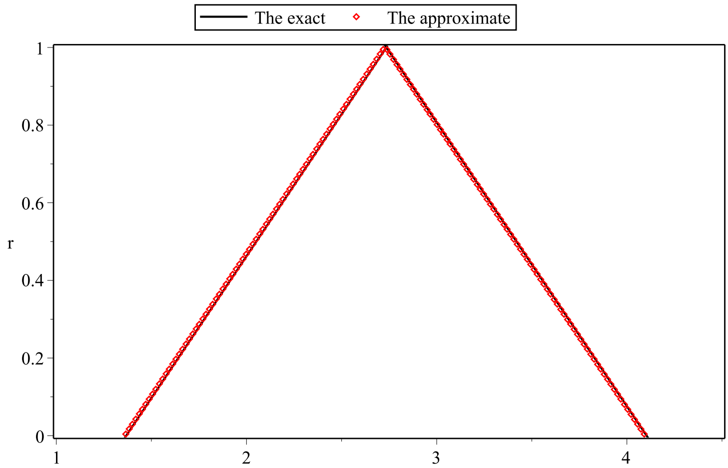

Figure 1: Exact solution and approximate solution for $t = 1$

The approximate fuzzy function calculated at the $24 - th$ iteration with $n = 51$. The following algorithm implements the trapezoidal rule using the MAPLE software.

Algorithm (5.2):

1. input $a, b, \lambda, k(s,t), \underline{f}(t,r), \overline{f}(t,r), n, m$
2. $h = \frac{b-a}{n}$
3. $t_0 = a, t_n = b$
4. For $i =,$ to n, compute $t_i = a + ih$
5. compute $\underline{S}_n{}^{(0)}(t,r) = \underline{f}(t,r)$,
$\underline{S}_n(r) = h\left[\frac{\underline{f}(a,r)+\underline{f}(b,r)}{2} + \sum_{i=1}^{n-1} \underline{f}(t_i,r)\right]$,
$\underline{S}_n{}^{(m)}(t,r) = \underline{f}(t,r) + \lambda \int_a^b k(s,t)\underline{S}_{n-1}{}^{(m-1)} ds$
6. compute $\overline{S}_n{}^{(0)}(t,r) = \overline{f}(t,r)$,
$\overline{S}_n(r) = h\left[\frac{\overline{f}(a,r)+\overline{f}(b,r)}{2} + \sum_{i=1}^{n-1} \overline{f}(t_i,r)\right]$,
$\overline{S}_n{}^{(m)}(t,r) = \overline{f}(t,r) + \lambda \int_a^b k(s,t)\overline{S}_{n-1}{}^{(m-1)} ds.$

Thus we obtain the following results:
$\underline{S}_{51}{}^{(0)}(t,r) = (r+1).(\exp(-t) + t - \sin t)$
$\underline{S}_{51}{}^{(1)}(t,r) = (r+1).[\exp(-t) + t] - 0.4311053600(r+1)\sin t$

$\underline{S}_{51}{}^{(2)}(t,r) = (r+1).[\exp(-t)+t] - 0.2143916095(r+1)\sin t$
$\underline{S}_{51}{}^{(3)}(t,r) = (r+1).[\exp(-t)+t] - (0.131837044r + 0.131837045)\sin t$
$\underline{S}_{51}{}^{(4)}(t,r) = (r+1).[\exp(-t)+t] - 0.1003888490(r+1)\sin t$
$\underline{S}_{51}{}^{(5)}(t,r) = (r+1).[\exp(-t)+t] - (0.088409027r + 0.088409028)\sin t$
$\underline{S}_{51}{}^{(6)}(t,r) = (r+1).[\exp(-t)+t] - 0.0838454550(r+1)\sin t$
$\underline{S}_{51}{}^{(7)}(t,r) = (r+1).[\exp(-t)+t] - 0.0821070160(r+1)\sin t$
$\underline{S}_{51}{}^{(8)}(t,r) = (r+1).[\exp(-t)+t] - 0.0814447775(r+1)\sin t$
$\underline{S}_{51}{}^{(9)}(t,r) = (r+1).[\exp(-t)+t] - 0.0811923050(r+1)\sin t$
$\underline{S}_{51}{}^{(10)}(t,r) = (r+1).[\exp(-t)+t] - 0.0810964055(r+1)\sin t$
$\underline{S}_{51}{}^{(11)}(t,r) = (r+1).[\exp(-t)+t] - 0.0810597975(r+1)\sin t$
$\underline{S}_{51}{}^{(12)}(t,r) = (r+1).[\exp(-t)+t] - 0.0810458515(r+1)\sin t$
$\underline{S}_{51}{}^{(13)}(t,r) = (r+1).[\exp(-t)+t] - 0.0810405390(r+1)\sin t$
$\underline{S}_{51}{}^{(14)}(t,r) = (r+1).[\exp(-t)+t] - 0.0810385145(r+1)\sin t$
$\underline{S}_{51}{}^{(15)}(t,r) = (r+1).[\exp(-t)+t] - (0.081037744r + 0.018037745)\sin t$
$\underline{S}_{51}{}^{(16)}(t,r) = (r+1).[\exp(-t)+t] - (0.081037450r + 0.081037452)\sin t$
$\underline{S}_{51}{}^{(17)}(t,r) = (r+1).[\exp(-t)+t] - (0.081037338r + 0.081037339)\sin t$
$\underline{S}_{51}{}^{(18)}(t,r) = (r+1).[\exp(-t)+t] - (0.081037296r + 0.081037297)\sin t$
$\underline{S}_{51}{}^{(19)}(t,r) = (r+1).[\exp(-t)+t] - (0.081037280r + 0.081037281)\sin t$
$\underline{S}_{51}{}^{(20)}(t,r) = (r+1).[\exp(-t)+t] - (0.081037274r + 0.081037275)\sin t$
$\underline{S}_{51}{}^{(21)}(t,r) = (r+1).[\exp(-t)+t] - 0.0810372715(r+1)\sin t$
$\underline{S}_{51}{}^{(22)}(t,r) = (r+1).[\exp(-t)+t] - 0.0810372705(r+1)\sin t$
$\underline{S}_{51}{}^{(23)}(t,r) = (r+1).[\exp(-t)+t] - 0.0810372705(r+1)\sin t$
$\underline{S}_{51}{}^{(24)}(t,r) = (r+1).[\exp(-t)+t] - 0.0810372705(r+1)\sin t$

and

$\overline{S}_{51}{}^{(0)}(t,r) = (3-r).(\exp(-t)+t-\sin t)$
$\overline{S}_{51}{}^{(1)}(t,r) = (3-r).[\exp(-t)+t] - (1.29331608 - 0.4311053600r)\sin t$
$\overline{S}_{51}{}^{(2)}(t,r) = (3-r).[\exp(-t)+t] - (0.64317483 - 0.2114391610r)\sin t$
$\overline{S}_{51}{}^{(3)}(t,r) = (3-r).[\exp(-t)+t] - (0.39551113 - 0.1318370445r)\sin t$
$\overline{S}_{51}{}^{(4)}(t,r) = (3-r).[\exp(-t)+t] - (0.30116655 - 0.1003888490r)\sin t$
$\overline{S}_{51}{}^{(5)}(t,r) = (3-r).[\exp(-t)+t] - (0.265227087 - 0.088409028r)\sin t$
$\overline{S}_{51}{}^{(6)}(t,r) = (3-r).[\exp(-t)+t] - (0.251536364 - 0.0838454545r)\sin t$
$\overline{S}_{51}{}^{(7)}(t,r) = (3-r).[\exp(-t)+t] - (0.246321045 - 0.0821070155r)\sin t$
$\overline{S}_{51}{}^{(8)}(t,r) = (3-r).[\exp(-t)+t] - (0.244334334 - 0.0814447775r)\sin t$
$\overline{S}_{51}{}^{(9)}(t,r) = (3-r).[\exp(-t)+t] - (0.243577158 - 0.0811925060r)\sin t$
$\overline{S}_{51}{}^{(10)}(t,r) = (3-r).[\exp(-t)+t] - (0.243289219 - 0.0810964060r)\sin t$

$$\overline{S}_{51}{}^{(11)}(t,r) = (3-r).[\exp(-t)+t] - (0.243179392 - 0.0805979750r)\sin t$$
$$\overline{S}_{51}{}^{(12)}(t,r) = (3-r).[\exp(-t)+t] - (0.243137552 - 0.0810458515r)\sin t$$
$$\overline{S}_{51}{}^{(13)}(t,r) = (3-r).[\exp(-t)+t] - (0.243121619 - 0.0810405390r)\sin t$$
$$\overline{S}_{51}{}^{(14)}(t,r) = (3-r).[\exp(-t)+t] - (0.243115544 - 0.0810385155r)\sin t$$
$$\overline{S}_{51}{}^{(15)}(t,r) = (3-r).[\exp(-t)+t] - (0.243113232 - 0.0810377450r)\sin t$$
$$\overline{S}_{51}{}^{(16)}(t,r) = (3-r).[\exp(-t)+t] - (0.243112352 - 0.0810374510r)\sin t$$
$$\overline{S}_{51}{}^{(17)}(t,r) = (3-r).[\exp(-t)+t] - (0.243112016 - 0.0103733850r)\sin t$$
$$\overline{S}_{51}{}^{(18)}(t,r) = (3-r).[\exp(-t)+t] - (0.243111886 - 0.0810372945r)\sin t$$
$$\overline{S}_{51}{}^{(19)}(t,r) = (3-r).[\exp(-t)+t] - (0.243111828 - 0.0810372790r)\sin t$$
$$\overline{S}_{51}{}^{(20)}(t,r) = (3-r).[\exp(-t)+t] - (0.243111822 - 0.0810372775r)\sin t$$
$$\overline{S}_{51}{}^{(21)}(t,r) = (3-r).[\exp(-t)+t] - (0.243111814 - 0.0810372715r)\sin t$$
$$\overline{S}_{51}{}^{(22)}(t,r) = (3-r).[\exp(-t)+t] - (0.243111811 - 0.0810372705r)\sin t$$
$$\overline{S}_{51}{}^{(23)}(t,r) = (3-r).[\exp(-t)+t] - (0.243111808 - 0.0810372705r)\sin t$$
$$\overline{S}_{51}{}^{(24)}(t,r) = (3-r).[\exp(-t)+t] - (0.243111808 - 0.0810372705r)\sin t.$$

Figure 2 compares the exact solution and the approximate solution for a fixed $t = 1$.

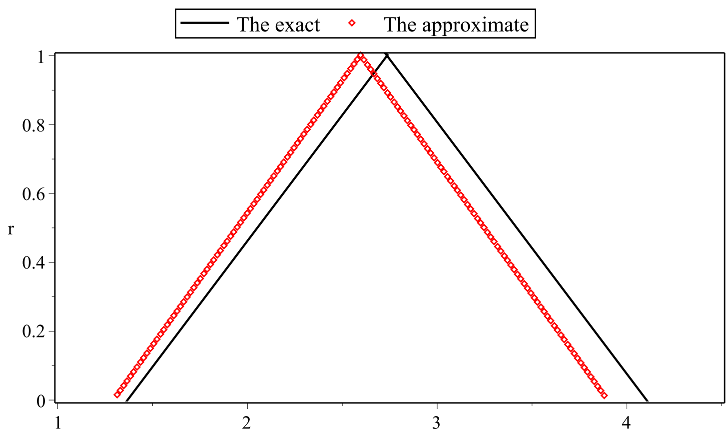

Figure 2: Exact solution and approximate solution for $t = 1$

r	$\underline{g}_{exact}$	$\overline{g}_{exact}$	$\underline{g}_{approximate}$	$\overline{g}_{approximate}$	$error = D(g_{exact}, g_{approximate})$
0.0	1.367879441	4.103638323	1.299688929	3.899066790	$2.045715329 \times 10^{-1}$
0.1	1.504667385	3.966850379	1.429657822	3.769097898	$1.977524810 \times 10^{-1}$
0.2	1.641455329	3.830062435	1.559626715	3.639129005	$1.909334300 \times 10^{-1}$
0.3	1.778243274	3.693274491	1.689595608	3.509160112	$1.841143791 \times 10^{-1}$
0.4	1.915031218	3.556486547	1.819564500	3.379191219	$1.772953280 \times 10^{-1}$
0.5	2.051819162	3.419698603	1.949533394	3.249222325	$1.704762770 \times 10^{-1}$
0.6	2.188607106	3.282910659	2.079502287	3.119253433	$1.636572250 \times 10^{-1}$
0.7	2.325395050	3.146122715	2.209471180	2.989284540	$1.568381739 \times 10^{-1}$
0.8	2.462182994	3.009334771	2.339440073	2.859315647	$1.500191229 \times 10^{-1}$
0.9	2.598970938	2.872546826	2.469408966	2.729346754	$1.432000720 \times 10^{-1}$
1.0	2.735758882	2.735758882	2.599377858	2.599377861	$1.363810240 \times 10^{-1}$

Table 2: The error resulted by algorithm (5.2) at $t = 1$

6. Conclusions

In this article, some numerical schemes, namely: the Taylor expansion method and the Trapezoidal method, have been investigated and implemented to approximate the solution of the fuzzy Fredholm integral equation of the second kind. A comparison between these methods shows the Taylor expansion method is more efficient than the Trapezoidal method. Moreover, it has been concluded in [11] that if the exact solution of the fuzzy Fredholm integral equation of the second kind is a polynomial then the metric error is zero, and this has also been justified in [2]. According to our test numerical example one can observe from Tables 1 and 2 and Figures 1 and 2 that the Taylor expansion method is more accurate than the Trapezoidal method.

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