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Numerical methods for computing the eigenvalues of linear fourth-order boundary-value problems

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Abstract

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Novel finite-difference methods are developed for approximating the eigenvalues of three types of linear, fourth-order, two-point, boundary-value problems. The fourth-order differential equation is transformed into a system of first-order equations and the numerical methods are derived by replacing the matrix exponential function in a recurrence relation by Padé approximants. Numerical results are obtained for a number of problems from the literature.

Keywords: Eigenvalue problems, fourth-order boundary-value problems, equivalent first-order problems, Padé approximants.

1. Introduction

Numerical methods are developed in the present paper for approximating the eigenvalues of linear, fourth-order, two-point, boundary-value problems. Such boundary-value problems occur in several areas of applied mathematics, physics, electrical engineering and mechanical engineering; most notably, such problems arise in the theory of vibrations of mechanical systems (see, for example, [1–3,5,6,10]).

The ordinary differential equations (ODEs) arising in linear, fourth-order, eigenvalue problems are typified by the following:

(I) the Euler–Bernoulli beam equation

$$[p(x)y''(x)]'' - \lambda s(x)y(x) = 0, \tag{1.1}$$

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(II) the second-type problem

$$[p(x)y''(x)]'' + [r(x) - \lambda s(x)]y(x) = 0, \tag{1.2}$$

and

(III) the general problem

$$[p(x)y''(x)]'' - [q(x)y'(x)]' + [r(x) - \lambda s(x)]y(x) = 0, \tag{1.3}$$

in which a prime denotes differentiation with respect to x.

It is clear that (1.2) is a special case of (1.3) when $q(x) = 0$ in the latter equation, and that (1.1) is a special case of (1.3) when $q(x) = r(x) = 0$. All three types have been considered in the literature, though it is probably the equation with $q(x) = r(x) = 0$ and $p(x) = 1$ which has received most attention. The numerical methods to be developed in the following section of the paper are applicable to all three types of ODE, together with one of two sets of boundary conditions. Numerical results are reported in the final section (Section 5) of the paper.

Consider, therefore, the linear, fourth-order, homogeneous, self-adjoint, two-point, boundary-value problem consisting of the differential equation

$$[p(x)y''(x)]'' - [q(x)y'(x)]' + [r(x) - \lambda s(x)]y(x) = 0, \quad a < x < b, \tag{1.4}$$

together with one of the following pairs of boundary conditions:

$$y(a) = y''(a) = y(b) = y''(b) = 0 \tag{1.5}$$

or

$$y(a) = y'(a) = y(b) = y'(b) = 0. \tag{1.6}$$

It is assumed that the real-valued functions $p(x)$, $q(x)$, $r(x)$ and $s(x)$ are continuous on the interval $[a, b]$ and satisfy the further conditions $p(x) \in C^2[a, b]$, $q(x) \in C^1[a, b]$, $p(x)$, $q(x)$, $s(x) > 0$ and $r(x) \geqslant 0$ for $x \in [a, b]$.

It is then known from [4, Theorem 10.1.2] and [7, Theorems 2.1 and 2.3] that the eigenvalues of (1.1)–(1.3) are real and positive.

Equation (1.4) can be transformed into a system of first-order differential equations. To this end let $w = w(x) = y'(x)$, $v = v(x) = y''(x)$ and $u = u(x) = y'''(x)$. Then (1.4) can be written as

$$\mathrm{D}Y(x) = Q(x)Y(x) + \lambda P(x)Y(x), \tag{1.7}$$

where

$$Y(x) = [u, v, w, y]^{\mathrm{T}}, \tag{1.8}$$

T denoting transpose, $\mathrm{D} = \mathrm{diag}\{\mathrm{d}/\mathrm{d}x\}$ and $Q = Q(x)$ and $P = P(x)$ are 4×4 matrices given by

$$Q(x) = \begin{bmatrix} \alpha(x) & \beta(x) & \gamma(x) & \delta(x) \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{1.9}$$

and

$$P(x) = \begin{bmatrix} 0 & 0 & 0 & \epsilon(x) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{1.10}$$

It then follows that, for the type (I) ODE in equation (1.1)

$$\alpha(x) = -2\frac{p'(x)}{p(x)}, \quad \beta(x) = -\frac{p''(x)}{p(x)}, \quad \gamma(x) = 0, \quad \delta(x) = 0, \quad \epsilon(x) = \frac{s(x)}{p(x)}, \tag{1.11}$$

for the type (II) equation (1.2)

$$\alpha(x) = -2\frac{p'(x)}{p(x)}, \quad \beta(x) = -\frac{p''(x)}{p(x)}, \quad \gamma(x) = 0, \quad \delta(x) = -\frac{r(x)}{p(x)},$$
$$\epsilon(x) = \frac{s(x)}{p(x)}, \tag{1.12}$$

and for the type (III) ODE in (1.3)

$$\alpha(x) = -2\frac{p'(x)}{p(x)}, \quad \beta(x) = \frac{q(x) - p''(x)}{p(x)}, \quad \gamma(x) = \frac{q'(x)}{p(x)}, \quad \delta(x) = -\frac{r(x)}{p(x)},$$
$$\epsilon(x) = \frac{s(x)}{p(x)}. \tag{1.13}$$

All numerical methods will be applied to the points $x_0, x_1, \ldots, x_N$ of the grid

$$G: \; a = x_0 < x_1 < x_2 < x_3 < \cdots < x_N < x_{N+1} = b,$$

obtained by discretizing the interval $x \in [a, b]$ into $N+1$ subintervals each of width $h = (b - a)/(N+1)$, where $N \geqslant 5$ is a positive integer; clearly, $h = x_{n+1} - x_n$, $n = 0, 1, \ldots, N$. The numerical methods will be developed by making approximations to the exponential term in the exact formula

$$Y(x+h) = \exp(hD)Y(x). \tag{1.14}$$

The theoretical solution of any of the three types of boundary-value problem at the grid points $x = x_m$, $m = 1, 2, \ldots, N$, at which the solution is sought, is obviously

$$Y(x_m) = [u(x_m), v(x_m), w(x_m), y(x_m)]^{\mathrm{T}} = [y'''(x_m), y''(x_m), y'(x_m), y(x_m)]^{\mathrm{T}}.$$

The solution of a convergent numerical method at the same grid point will be denoted by

$$Y_m = [u_m, v_m, w_m, y_m]^{\mathrm{T}},$$

where u_m, v_m, w_m, y_m denote the associated approximations to $y'''(x)$, $y''(x)$, $y'(x)$, $y(x)$, respectively, at the point $x = x_m$, $m = 1, 2, \ldots, N$. Obviously, the subscripts 0 and $N+1$ in all notations refer to boundary points. It will be convenient to define a vector $\tilde{Y}$ of order $4(N+1)$ by

$$\tilde{Y} = [Y_1^{\mathrm{T}}, Y_2^{\mathrm{T}}, \ldots, Y_N^{\mathrm{T}}, Y_{N+1}^{\mathrm{T}}]^{\mathrm{T}}. \tag{1.15}$$

2. A second-order method

Using the (1, 1) Padé approximant to the exponential term in equation (1.14) leads to the second-order equation

$$[I - \tfrac{1}{2}h\mathrm{D}]Y(x+h) = [I + \tfrac{1}{2}h\mathrm{D}]Y(x) + \mathrm{O}(h^3), \tag{2.1}$$

in which I is the identity matrix of order four. Applying (1.7) then gives

$$\begin{aligned}&[I - \tfrac{1}{2}hQ(x+h)]Y(x+h) - [I + \tfrac{1}{2}hQ(x)]Y(x)\\ &\quad = \lambda[\tfrac{1}{2}hP(x+h)Y(x+h) + \tfrac{1}{2}hP(x)Y(x)] + \mathrm{O}(h^3),\end{aligned} \tag{2.2}$$

in which P, Q and Y are defined by (1.10), (1.9) and (1.8), respectively.

Applying (2.2) to each of the grid points x_k, $k = 0, 1, \ldots, N$, of G gives

$$A_{k+1}Y_{k+1} + B_kY_k = \lambda[E_{k+1}Y_{k+1} + F_kY_k], \quad k = 0, 1, 2, \ldots, N, \tag{2.3}$$

in which

$$A_{k+1} = I - \tfrac{1}{2}hQ_{k+1}, \qquad B_k = -I - \tfrac{1}{2}hQ_k, \qquad E_{k+1} = \tfrac{1}{2}hP_{k+1}, \qquad F_k = \tfrac{1}{2}hP_k, \tag{2.4}$$

$k = 0, 1, \ldots, N$, are all square matrices of order four with $P_k = P(x_k)$ and $Q_k = Q(x_k)$ defined by (1.10) and (1.9), respectively.

It is clear that applying (2.3) with $k = 0, 1, \ldots, N$ results in the embedding of the matrices in A_{k+1} and B_k in a block matrix $\tilde{A}$ given by

$$\tilde{A} = \begin{bmatrix} A_1 & & & & B_0 \\ B_1 & A_2 & & & \\ & B_2 & A_3 & & \\ & & \ddots & \ddots & \\ & & & B_N & A_{N+1} \end{bmatrix}, \tag{2.5}$$

while the matrices E_{k+1} and F_k are embedded in a block matrix $\tilde{B}$ given by

$$\tilde{B} = \begin{bmatrix} E_1 & & & & F_0 \\ F_1 & E_2 & & & \\ & F_2 & E_3 & & \\ & & \ddots & \ddots & \\ & & & F_N & E_{N+1} \end{bmatrix}. \tag{2.6}$$

The system of equations described by (2.3) can thus be written as the generalized eigenvalue problem

$$\tilde{A}\tilde{Y} = \lambda \tilde{B}\tilde{Y}. \tag{2.7}$$

Any change in the boundary conditions (1.5) or (1.6) will be reflected in the submatrices A_{N+1}, B_0, E_{N+1} and F_0 in (2.5) and (2.6).

3. A fourth-order method

Using, now, the (2, 2) Padé approximant to the exponential term in equation (1.14) leads to the fourth-order equation

$$\left[I-\tfrac{1}{2}h\mathrm{D}+\tfrac{1}{12}h^2\mathrm{D}^2\right]Y(x+h)=\left[I+\tfrac{1}{2}h\mathrm{D}+\tfrac{1}{12}h^2\mathrm{D}^2\right]Y(x)+\mathrm{O}(h^5). \tag{3.1}$$

Differentiating (1.7) gives

$$\mathrm{D}^2Y(x)=Q^*(x)Y(x)+\lambda P^*(x)Y(x), \tag{3.2}$$

where $Q^*(x)$ and $P^*(x)$ are given by

$$Q^*(x)=\mathrm{D}Q(x)+Q^2(x) \tag{3.3}$$

and

$$P^*(x)=Q(x)P(x)+\mathrm{D}P(x)+P(x)Q(x). \tag{3.4}$$

Then, using (1.7) and (3.2) in (3.1) gives

$$\begin{aligned}&\left[I-\tfrac{1}{2}hQ(x+h)+\tfrac{1}{12}h^2Q^*(x+h)\right]Y(x+h)-\left[I+\tfrac{1}{2}hQ(x)+\tfrac{1}{12}h^2Q^*(x)\right]Y(x)\\&\quad=\lambda\left[\left\{\tfrac{1}{2}hP(x+h)-\tfrac{1}{12}h^2P^*(x+h)\right\}Y(x+h)+\left\{\tfrac{1}{2}hP(x)+\tfrac{1}{12}h^2P^*(x)\right\}Y(x)\right]\\&\qquad+\mathrm{O}(h^5),\end{aligned} \tag{3.5}$$

which, when applied to the points x_k, $k=0, 1,\ldots, N$, of the grid G, gives an equation of the form (2.3) with, now,

$$A_{k+1}=I-\tfrac{1}{2}hQ_{k+1}+\tfrac{1}{12}h^2Q^*_{k+1}, \tag{3.6}$$

$$B_k=-I-\tfrac{1}{2}hQ_k-\tfrac{1}{12}h^2Q^*_k, \tag{3.7}$$

$$E_{k+1}=\tfrac{1}{2}hP_{k+1}-\tfrac{1}{12}h^2P^*_{k+1} \tag{3.8}$$

and

$$F_k=\tfrac{1}{2}hP_k+\tfrac{1}{12}h^2P^*_k. \tag{3.9}$$

Clearly, $P^*_k=P^*(x_k)$ and $Q^*_k=Q^*(x_k)$.

The system of equations described by (2.3) with (3.6)–(3.9) can thus be written as the generalized eigenvalue problem (2.7), in which the block matrices $\tilde{A}$ and $\tilde{B}$ are compiled using the submatrices given in (3.6)–(3.9).

4. Computing the eigenvalues

The smallest positive real eigenvalue of the generalized eigenvalue problem (2.7) may be computed using the following procedure.

Firstly, (2.7) is transformed to the usual eigenvalue problem

$$\tilde{C}\tilde{Y}=\Lambda\tilde{Y}, \tag{4.1}$$

where $\tilde{C}=\tilde{A}^{-1}\tilde{B}$ and $\Lambda=\lambda^{-1}$, $\lambda>0$. Secondly, the Power Method is used to find the largest eigenvalue of $\tilde{C}$ in (4.1) and its corresponding eigenvector. In fact, the power method is used here without having to find the inverse of block matrix $\tilde{A}$.

To this end, let $\tilde{Y}_0$ be an arbitrary column vector with $4N+4$ elements: it is convenient to take $\tilde{Y}_0=[1, 1,\ldots, 1]^T$, the unit vector. The equation $\tilde{Z}_i=\tilde{C}\tilde{Y}_i$, $i=0, 1, 2,\ldots,$ is equivalent to $\tilde{A}\tilde{Z}_i=\tilde{B}\tilde{Y}_i$ and $\tilde{Z}_i$ may therefore be computed by writing $\tilde{A}=\tilde{L}\tilde{U}$ and finding the decomposition matrices $\tilde{L}$ and $\tilde{U}$ of $\tilde{A}$ using, say, Doolittle's method.

Now let $\tilde{W}_i=\tilde{B}\tilde{Y}_i$ and compute the vector $\tilde{V}_i$ defined by the equation $\tilde{L}\tilde{V}_i=\tilde{W}_i$; then solve for $\tilde{Z}_i$ using the equation $\tilde{U}\tilde{Z}_i=\tilde{V}_i$. Define, next, $\alpha_i=\|\tilde{Z}_i\|_\infty$ and update by normalizing the vector $\tilde{Z}_i$ to give $\tilde{Y}_{i+1}=\alpha_i^{-1}\tilde{Z}_i$.

Convergence of the sequence $\{\alpha_i\}$ is to Λ, the largest eigenvalue of the matrix $\tilde{C}$ in (4.1). The smallest eigenvalue λ is therefore the value to which the sequence $\{\alpha_i^{-1}\}$ converges.

To compute all the eigenvalues of the matrix $\tilde{C}$ in (4.1), note first of all that $\tilde{A}^{-1}=(\tilde{L}\tilde{U})^{-1}=\tilde{U}^{-1}\tilde{L}^{-1}$. Then $\tilde{L}^{-1}$ can be found by solving the system $\tilde{L}\tilde{Z}_j=\tilde{e}_j$, $j=1, 2,\ldots,4N+4$, for $\tilde{Z}_j$, where $\tilde{e}_j$ is the column vector of order $4N+4$, the elements of which are all zero except the jth element which is unity. The vector $\tilde{Z}_j$ will be the jth column of matrix $\tilde{L}^{-1}$.

Similarly, $\tilde{U}^{-1}$ can be computed by solving the system $\tilde{U}\tilde{Z}_j=\tilde{e}_j$, $j=1, 2,\ldots,4N+4$, for $\tilde{Z}_j$ which, now, is the jth column of the matrix $\tilde{U}^{-1}$.

Having determined $\tilde{L}^{-1}$ and $\tilde{U}^{-1}$, the eigenvalue problem (4.1) may be rewritten as the equivalent problem

$$\tilde{U}^{-1}\tilde{L}^{-1}\tilde{B}\tilde{Y}=\Lambda\tilde{Y}. \tag{4.2}$$

All the eigenvalues of (4.2) can be determined using the NAG (Numerical Algorithms Group) FORTRAN subroutine F02AFF which gives all the eigenvalues of the real matrix $\tilde{C}=\tilde{A}^{-1}\tilde{B}=\tilde{U}^{-1}\tilde{L}^{-1}\tilde{B}$.

5. Numerical experiments

Four different eigenvalue problems from the literature [8,9] are considered in the following numerical examples. In each case the smallest eigenvalue is determined for a series of values of N. In these problems the exact values of the eigenvalues cannot be obtained by analytical methods and so it is assumed (as in [8,9]) that the computed value of the smallest eigenvalue obtained using the largest N is the exact (smallest) eigenvalue λ_e. Comparisons should then be made on the relative error RE of the value λ_N, obtained using some other value of N, which is computed from the equation

$$\mathrm{RE}=\left|\frac{\lambda_e-\lambda_N}{\lambda_e}\right|. \tag{5.1}$$

Example 5.1 (Usmani and Isa [8]). This example consists of the differential equation

$$\left[(1+x^2)y''\right]''-\left[(1+x^2)y'\right]'+\left[(1+x)^{-2}-\lambda(1+x)^4\right]y=0, \quad 0<x<1, \tag{5.2}$$

subject to the boundary conditions

$$y(0)=y''(0)=y(1)=y''(1)=0. \tag{5.3}$$

In [8], Usmani and Isa used a method with second-order convergence to calculate their results which are reproduced in Table 5.1. Results obtained using the second-order method of Section

Table 5.1
Observed relative errors for $h = 2^{-m}$ of Example 5.1 in [8]

m	N	λ	Relative errors
3	7	24.634681	$2.448 \cdot 10^{-2}$
4	15	25.085489	$6.068 \cdot 10^{-3}$
5	31	25.199984	$1.497 \cdot 10^{-3}$
6	63	25.228721	$3.563 \cdot 10^{-4}$
7	127	25.235913	$7.125 \cdot 10^{-5}$
8	255	25.237711	0.0

2 and the fourth-order method of Section 3 are listed in Tables 5.2 and 5.3, respectively. on all tables, the first two columns give m and N, where $h = (b-a)/(N+1) = 1/(N+1)$ and $N+1 = 2^m$.

All computations were performed in double-precision arithmetic using a Pyramid 9820 computer. The columns headed U, S and R in Tables 5.2 and 5.3 give, in seconds, the user time, the system time and, in minutes and seconds, the real time, respectively. The CPU-time is the sum of U and S, each of which is rounded to one decimal place. The final columns of each table give the smallest positive eigenvalue and the associated relative error calculated for the different values of N.

It is seen from the three tables that the numerical results obtained for large values of N, using the fourth-order method of Section 3, are closer to the results of [8] than the correspond-

Table 5.2
Second-order method for computing the smallest eigenvalue of Example 5.1

m	N	S	U	R	λ	Relative error
3	7	0.0	0.0	0:0	26.840469032602	$6.3477247447894 \cdot 10^{-2}$
4	15	0.0	0.0	0:0	25.627155371202	$1.5403144445101 \cdot 10^{-2}$
5	31	0.0	0.1	0:0	25.334813477918	$3.8199283840945 \cdot 10^{-3}$
6	63	0.0	0.2	0:0	25.262391076194	$9.5039669500796 \cdot 10^{-4}$
7	127	0.0	0.5	0:0	25.244328279975	$2.3470976789564 \cdot 10^{-4}$
8	255	0.0	1.1	0:1	25.239813436483	$5.5821935357914 \cdot 10^{-5}$
9	511	0.2	2.4	0:3	25.238685232680	$1.1120068430071 \cdot 10^{-5}$
10	1023	0.2	4.9	0:3	25.238404579894	0.0

Table 5.3
Fourth-order method for computing the smallest eigenvalue of Example 5.1.

m	N	U	S	R	λ	Relative error
3	7	0.0	0.0	0:0	25.243296971413	$1.9756930366577 \cdot 10^{-4}$
4	15	0.1	0.0	0:0	25.238625220557	$1.2463774231586 \cdot 10^{-5}$
5	31	0.2	0.0	0:0	25.238330362854	$7.8083288812536 \cdot 10^{-7}$
6	63	0.4	0.0	0:0	25.238311888279	$4.8827673682084 \cdot 10^{-8}$
7	127	1.1	0.1	0:1	25.238310732895	$3.0486986268841 \cdot 10^{-9}$
8	255	2.1	0.1	0:2	25.238310660732	$1.8943424606732 \cdot 10^{-10}$
9	511	4.5	0.1	0:4	25.238310656048	$3.8431480220424 \cdot 10^{-12}$
10	1023	9.3	0.1	0:11	25.238310655951	0.0

Table 5.4
Observed relative errors for $h = 2^{-m}$ of Example 5.2 in [8]

m	N	λ	Relative error
3	7	19.548553	$2.551 \cdot 10^{-2}$
4	15	19.921847	$6.294 \cdot 10^{-3}$
5	31	20.016196	$1.551 \cdot 10^{-3}$
6	63	20.039847	$3.691 \cdot 10^{-4}$
7	127	20.045764	$7.380 \cdot 10^{-5}$
8	255	20.047244	0.0

ing results obtained using the second-order method of Section 2. Comparison of Tables 5.2 and 5.3 also reveals that the fourth-order method of Section 3 is more expensive to implement than the second-order method of Section 2.

Example 5.2 (Usmani and Isa [8]). Here, the differential equation is

$$(e^x y'')'' - (e^x y')' + (\sin x - \lambda \cos x)y = 0, \quad 0 < x < 1, \tag{5.4}$$

and the boundary conditions are given by (5.3).

The results of [8] are reproduced in Table 5.4, while results obtained using the methods of Sections 2 and 3 are given in Tables 5.5 and 5.6, respectively. The present authors suspect that

Table 5.5
Second-order method for computing the smallest eigenvalue of Example 5.2

m	N	U	S	R	λ	Relative error
3	7	0.0	0.0	0:0	208.94307752710	$4.2221104130598 \cdot 10^{-2}$
4	15	0.0	0.0	0:0	202.52962876664	$1.0230421656111 \cdot 10^{-2}$
5	31	0.1	0.0	0:1	200.98652615748	$2.5333295861572 \cdot 10^{-3}$
6	63	0.3	0.2	0:1	200.60441678082	$6.2734418020581 \cdot 10^{-4}$
7	127	0.7	0.0	0:1	200.50911416626	$1.5196879400573 \cdot 10^{-4}$
8	255	1.4	0.1	0:3	200.48530640792	$3.3214210328536 \cdot 10^{-5}$
9	511	3.2	0.4	0:4	200.47935461822	$1.1120068430071 \cdot 10^{-6}$
10	1023	6.8	0.3	0:7	200.47864766795	0.0

Table 5.6
Fourth-order method for computing the smallest eigenvalue of Example 5.2

m	N	U	S	R	λ	Relative error
3	7	0.0	0.0	0:0	200.47949572450	$1.0615265780167 \cdot 10^{-5}$
4	15	0.1	0.0	0:0	200.47748680264	$5.9457424761433 \cdot 10^{-7}$
5	31	0.2	0.0	0:0	200.47737483909	$3.6089510269122 \cdot 10^{-8}$
6	63	0.4	0.0	0:0	200.47736805264	$2.2380581832238 \cdot 10^{-9}$
7	127	0.9	0.0	0:1	200.47736763152	$1.3747181171198 \cdot 10^{-10}$
8	255	2.1	0.2	0:2	200.47736760529	$6.6342487059501 \cdot 10^{-12}$
9	511	4.5	0.2	0:5	200.47736760380	$7.9791728779810 \cdot 10^{-13}$
10	1023	9.4	0.2	0:9	200.47736760396	0.0

Table 5.7
Observed relative errors for $h = 2^{-m}$ of Example 5.3 in [9]

m	N	λ	Relative error [9, (2.8)]	λ	Relative error [9, (3.5)]
3	7	22.187	$2.557 \cdot 10^{-2}$	22.746419	$3.358 \cdot 10^{-4}$
4	15	22.610	$6.352 \cdot 10^{-3}$	22.753574	$2.129 \cdot 10^{-5}$
5	31	22.718	$1.586 \cdot 10^{-3}$	22.754027	$1.358 \cdot 10^{-6}$
6	63	22.745	$3.962 \cdot 10^{-4}$	22.754056	$1.078 \cdot 10^{-7}$
7	127	22.752	$9.907 \cdot 10^{-5}$		
8	255	22.753	$2.480 \cdot 10^{-5}$		

the eigenvalues given in Table 5.4 contain typographical errors and are incorrect by a factor of 10, as the results given in Tables 5.5 and 5.6 were obtained using different methods. All the observations made on the results of Example 5.1 are applicable to those of Example 5.2.

Example 5.3 (Usmani and Sakai [9]). In this example, the differential equation is given by

$$\left[(1+x^2)y''\right]'' + \left[(1+x^2)^{-1} - \lambda(1+x)^4\right]y = 0, \quad 0 < x < 1, \tag{5.5}$$

and the associated boundary conditions are given by (5.3) once again. Unlike the differential equations (5.2) and (5.4) which are of the form (1.3), equation (5.5) is an example of the second-type problem (1.2).

The numerical results obtained in [9] using the method given in [9, (2.8)], on grids for which $h = 2^{-m}$ with $Nh = 1$ and $N + 1 = 2^m$, $m = 3, 4, \ldots, 8$, are given in Table 5.7. Numerical results reported in [9] for the numerical method given in equation (3.5) of that paper, using $m = 3$–6 only, are also given in Table 5.7. The present authors suspect that equation [9, (4.1)] also contains a typographical error and that the differential equation used in [9] should be

$$\left[(1+x^2)y''\right]'' + \left[(1+x)^{-2} - \lambda(1+x)^4\right]y = 0, \quad 0 < x < 1. \tag{5.6}$$

It was seen, when solving the eigenvalue problems in Examples 5.2 and 5.3 of the present paper, that the fourth-order method of Section 3 gives results which are closer to published results than the second-order method of Section 2. Accordingly, only results obtained using the fourth-order method were obtained, and these are given in Table 5.8. It is noted once more that the relative errors obtained using this method are superior to those reported in [9].

Table 5.8
Fourth-order method for computing the smallest eigenvalue of Example 5.3

m	N	U	S	R	λ	Relative error
3	7	0.0	0.0	0:0	22.815360699869	$4.5031836880982 \cdot 10^{-5}$
4	15	0.0	0.0	0:0	22.814373591854	$1.7648257093494 \cdot 10^{-6}$
5	31	0.1	0.0	0:0	22.814337002830	$1.6105217470752 \cdot 10^{-7}$
6	63	0.3	0.0	0:0	22.814333780997	$1.9832488318272 \cdot 10^{-8}$
7	127	0.6	0.1	0:0	22.814333514902	$8.1689874864566 \cdot 10^{-9}$
8	255	1.5	0.0	0:1	22.814333321330	$3.1567892744278 \cdot 10^{-10}$
9	511	3.2	0.1	0:3	22.814333328530	0.0

Table 5.9
Observed relative errors for $h = 2^{-m}$ of Example 5.4 in [9]

m	N	λ	Relative error [9, (2.8)]	λ	Relative error [9, (3.5)]
3	7	176.641	$2.664 \cdot 10^{-2}$	181.244637	$5.564 \cdot 10^{-4}$
4	15	180.159	$6.588 \cdot 10^{-3}$	181.339089	$3.529 \cdot 10^{-5}$
5	31	181.048	$1.642 \cdot 10^{-3}$	181.345093	$2.175 \cdot 10^{-6}$
6	63	181.271	$4.103 \cdot 10^{-4}$	181.345470	$9.728 \cdot 10^{-8}$
7	127	181.327	$1.025 \cdot 10^{-4}$		
8	255	181.341	$2.560 \cdot 10^{-5}$		

Example 5.4 (Usmani and Sakai [9]). In this final example the differential equation is also of the second type (1.2) and is given by

$$(e^x y'')'' + (\sin x - \lambda \cos x)y = 0, \quad 0 < x < 1, \tag{5.7}$$

and the boundary conditions are given by (5.3).

The results obtained in [9] using its formulae (2.8) and (3.5) are reproduced in Table 5.9 where the values of h, N and m are the same as those used in Example 5.3. The results obtained using the fourth-order method of Section 3 are listed in Table 5.10.

As in Examples 5.1–5.3, the smallest eigenvalue obtained using the method of the present paper is seen to decrease as N is increased. This is in contrast to the numerical results reported in [8,9] where the smallest eigenvalue was seen to increase as N was increased. Using a large value of N, the results obtained using the methods of the present paper are in good agreement with those reported in [8,9].

6. Summary

Two finite-difference methods, one second-order convergent and one fourth-order convergent, have been developed and tested for approximating the eigenvalues of three types of linear, fourth-order, two-point, boundary-value problems.

The fourth-order ordinary differential equation was transformed into a system of first-order equations and the numerical methods were derived by replacing the matrix exponential function in a recurrence relation, by one second-order and one fourth-order Padé approximant.

Table 5.10
Fourth-order method for computing the smallest eigenvalue of Example 5.4

m	N	U	S	R	λ	Relative error
3	7	0.0	0.0	0:0	181.34804763802	$1.4071915540592 \cdot 10^{-5}$
4	15	0.1	0.0	0:0	181.34564304647	$8.1218973413399 \cdot 10^{-7}$
5	31	0.2	0.0	0:0	181.34550479478	$4.9823459802312 \cdot 10^{-8}$
6	63	0.4	0.0	0:0	181.34549637367	$3.3866294213425 \cdot 10^{-9}$
7	127	1.0	0.0	0:1	181.34549574049	$1.0493783619836 \cdot 10^{-10}$
8	255	2.2	0.0	0:2	181.345495618.91	$7.7537076759171 \cdot 10^{-10}$
9	511	4.7	0.1	0:5	181.34549575952	0.0

Numerical results obtained using the two methods were compared with results reported in the literature.

References

[1] R.E.D. Bishop and W.G. Price, *Hydroelasticity of Ships* (Cambridge Univ. Press, Cambridge, 1979).
[2] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* (Clarendon Press, Oxford, 1961); reprinted: (Dover, New York, 1981).
[3] G.M.L. Gladwell, Qualitative properties of vibrating systems, *Proc. Roy. Soc. London Ser. A* **401** (1985) 299–315.
[4] G.M.L. Gladwell, *Inverse Problems in Vibration* (Martinus Nijhoff, Dordrecht, 1986).
[5] D.J. Gorman, *Free Vibration Analysis of Beams and Shafts* (Wiley, New York, 1975).
[6] F.B. Hildebrand, *Advanced Calculus for Applications* (Prentice-Hall, Englewood Cliffs, NJ, 1964).
[7] R.A. Usmani, Finite difference methods for computing eigenvalues of fourth-order boundary-value problems, *Internat. J. Math. Math. Sci.* **9** (1986) 137–143.
[8] R.A. Usmani and M. Isa, Numerical methods for approximating eigenvalues of boundary-value problems, *Internat. J. Math. Math. Sci.* **9** (1986) 567–575.
[9] R.A. Usmani and M. Sakai, Two new finite difference methods for computing eigenvalues of a fourth-order linear boundary-value problem, *Internat. J. Math. Math. Sci.* **10** (1987) 525–530.
[10] J.R. Vinson, *The Behavior of Thin Walled Structures: Beams, Plates and Shells* (Kluwer Academic Publishers, Dordrecht, 1989).