Novel Fractional Wavelet Transform with Closed-Form Expression

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***Abstract*—A new wavelet transform (WT) is introduced based on the fractional properties of the traditional Fourier transform. The new wavelet follows from the fractional Fourier order which uniquely identifies the representation of an input function in a fractional domain. It exploits the combined advantages of WT and fractional Fourier transform (FrFT). The transform permits the identification of a transformed function based on the fractional rotation in time-frequency plane. The fractional rotation is then used to identify individual fractional daughter wavelets. This study is, for convenience, limited to one-dimension. Approach for discussing two or more dimensions is shown.**

Keyword—Fractional Fourier transform; wavelet; fractional wavelet transform

I. Introduction

An introduction has been given to a new family of wavelets that are formed from the fractional Fourier order of the Fourier transform [1-5]. The fractional order of the Fourier transform is discussed based on discrete Fourier transform (DFT) as the fractional Fourier transform (FrFT) [6, 7] which is believed to be related to the chirp-Fourier transform [7, 8]. The chirp signal is highly concentrated in the fractional domain and a time delay leads to a fractional shift making the FrFT an efficient tool for separating the chirp signal [2]. In fact, the property of the FrFT tool enables that such delays choreographing into/as noise can be effectively filtered off [8]. Meanwhile, there are other possible characterizations of the fractional domain based tools that can be derived from the fractional Fourier tool. This was introduced as fractional wave packet transform (FRWPT) in [3]. The FRWPT was aimed at combining the advantages of the WT and FrFT, but this transform is computationally expensive [2]. In the recent times, this relationship has received wider attention in the discussion of wavelet families based on the fractional order of the DFT such as in [1, 9] and in [2]. It is called the fractional wavelet transform (FrWT). It is hoped that the proposed FrWT circumvents the computational cost available in [3]. Although the new wavelet transform discussed in [3] stemmed from the parent impulse filter property of the wavelet function, we approach the problem in a new fashion. For instance, in this work we extend this novelty into discussing wavelet transform using the quadratic phase function, as an example, earlier mentioned in [1] and based on FrFT by exploiting the dilation and translation properties of the mother wavelets demonstrated in [10].

At the moment, other studies have followed different methods to showing the exact closed-form expression for wavelet transform, for instance, by using raised cosine function [11]. The exact closed form expression for discrete wavelet transform based on FrFT is derived in this study. The complexity of the transform within a novel family of wavelet proposed here is also described. The computational gain exhibited in this new design is well spelt out and stressed. This would however revive the interest in deploying wavelet in signal processing, for instance.

We have organized the remaining parts of this paper as: In Section II, we familiarize the reader with the basic wavelet theory, then the proposed wavelet in Section III and the closed-form expression for the proposed wavelet is described in Section IV. The conclusion is presented in Section V.

II. Traditional Wavelet Theory

Wavelets are orthonormal functions derived from the parent scaling functions. For instance, consider an input signal $f(t)$, that modulates the transforming function, or scaling function, $\varphi(t)$. There are narrowband functions $\psi(t)$ derivable from $\varphi(t)$, which are orthogonal wavelets useful in the design of multicarrier systems. By the Fourier relation and Parseval's theory, the signal for band-limited case can be periodic with β, $-2\pi \leq \beta \leq 2\pi$, so that if $\psi_{l,m}(t)$ belongs to a set of orthonormal functions, then;

$$\int \psi_{l,m}(t)\psi_{k,n}(t)dt = \delta(l-k)\delta(m-n) \tag{1}$$

where $\delta(\bullet)$ is a Dirac delta. Equation (1) defines a simple orthogonality condition between two daughter wavelets. Since $\psi(t)$ is obtained from the decomposition of $\varphi(t)$, we can express the relationship of the input signal with $\varphi(t)$ in discrete form as [12];

$$S_{DWT} = \sum_{m=0}^{M-1} f(m)\varphi_m(t) \tag{2}$$

where M is the length of the characteristic filter. $f(m)$ is the discrete equivalent of $f(t)$. The mother wavelet has a clear relation to the filters;

$$\psi(t) = \sqrt{2}\sum_{m} g(m)\varphi(2t-m) \tag{3}$$

where $g(m)$ is a high-pass filter (HPF) and can be directly derived or constructed from a low-pass filter (LPF) that comes from the parent scaling function as;

$$\varphi(t) = \sqrt{2}\sum_{m} h(m)\varphi(2t-m) \qquad (4)$$

where $h(m)$ is the LPF and $\varphi(t)$ is the scaling function. Thus, as an alternative to modulating the input symbols by the sinusoids, these half-band filters can be used. The high-pass filters construct the detail coefficient part of the signal while the low-pass filters construct the approximate coefficient part of the signal. The high-pass filter can be formed from the low-pass filter as:

$$g(n) = (-1)^n h(M+1-n) \qquad (5)$$

where M is the length of the filter and n is the prevailing filter coefficient index. Both the high-pass and low-pass filters constitute the filter bank [13] required in multiresolution analysis (MRA). In signal processing for example, the multiplexing/processing function can be equivalently used orthogonal basis function such as [14]:

$$\varphi_{m,n}(t) = \begin{cases} 1 & n = m \\ 0 & elsewhere \end{cases}$$

where m and n are scales and shifts respectively. $\varphi_{m,n}(t)$ represents the complex orthogonal DWT basis function similar to the traditional multicarrier system.

III. Proposed Fractional Wavelet Transform

It is a common knowledge that when a mother wavelet is translated and dilated, the daughter wavelets are born. The shifting and translation parameters of the father wavelet (or scaling function) can be well represented and approximated respectively, each of which gives rise to a uniquely different family of wavelets (see [10]).

Earlier, [15] identified discrete wavelets for any family by approximating the shift and translation parameters to discrete coefficients. Alternatively, the DFT roots can as well be exploited to define a new family of wavelets, namely, fractional wavelet transform. The fractional orders of the DFT define a uniquely different FrFTs which must also identify daughter wavelets [1] consequent on the DFT roots [16] that characterize the FrFT order.

Now, let us recall the definition of wavelet as represented in [1, 3]; for instance, let the mother wavelet be defined as,

$$\psi(t) = e^{i\pi t^2} \qquad (6)$$

But daughter wavelets are translated and shifted parts of Equation 6. So, let the daughter wavelets be defined as:

$$\psi_{(k,\tau)} = \frac{1}{\sqrt{k}}\psi\left(\frac{t-\tau}{k}\right) \qquad (7)$$

where τ and k are shifts and scale parameters respectively. These parameters are defined as $k = 2^d$ and $\tau = 2^d n$ for traditional discrete wavelets [15].

Equation 7 (in discrete sense) becomes $\psi_{d,n} = 2^{-d/2}\psi(2^{-d}t-n)$ where d and n are equivalent shift and scale parameters respectively. Let Equation 7 be defined in terms of Equation 6 as:

$$\psi_{(k,\tau)}(t,k) = \frac{1}{\sqrt{k}}\exp\left[i\pi\left(\frac{t-\tau}{k}\right)^2\right] \qquad (8)$$

If we consider a signal $f(t)$ to be transformed by the wavelet transform, then the resulting transformation can be expressed as:

$$x(\tau,k) = \frac{1}{\sqrt{k}}\int_{-\infty}^{+\infty}\exp\left[i\pi\left(\frac{t-\tau}{k}\right)^2\right]f(\tau)\,d\tau \qquad (9)$$

Notice that for unit impulse [17], in other words, for a maximum of unit amplitude input signals:

$$\int_{-\infty}^{+\infty} f(t)\delta(t-\tau)d\tau = f(\tau)$$

Equation 9 is necessarily a Fourier transform of $f(t)$ that is shifted and translated by τ and k respectively. For a rotation angle α which defines the fractional Fourier order of *a-fractional* rotation, $0 < |\alpha| < \pi$ (i.e. $0 < |a| < 2$), then the fractional Fourier transform when α is not a multiple of π-rad can be expressed as [1, 3, 16]:

$$B_a(t,\tau) = \frac{\exp\left[-i\{\pi\hat{\alpha}/4-\alpha/2\}\right]}{\sqrt{\sin\alpha}} \times \exp\left[i\pi(t^2\cot\alpha - 2t\tau\sec\alpha + \tau^2\cot\alpha)\right] \qquad (10)$$

where $\hat{\alpha} = \mathrm{sgn}(\sin\alpha)$ and $\alpha = \frac{a\pi}{2}$. Thus, there exist a unique fractional Fourier transform for every order a. From Equation 10, let $u = t\sec\alpha$ such that,

$$x(u,\tau) = f(t) = f\left(\frac{u}{\sec\alpha}\right) = A_\alpha D_\alpha\int_{-\infty}^{+\infty}\exp\left[i\pi\left(\frac{u-\tau}{\tan^{1/2}\alpha}\right)^2\right]f(\tau)d\tau \qquad (11)$$

where,

$$A_\alpha = \frac{\exp\left[-i(\pi\hat{\alpha}/4-\alpha/2)\right]}{\sqrt{\sin\alpha}} \text{ and } D_\alpha = \exp(-i\pi u^2\sin^2\alpha)$$

Figure 1 shows an asymptotic representation of the behaviour of DFT roots which implies that the possible Fourier roots that govern the fractional Fourier order can never be zero. Although at a = 1, the traditional DFT is obtained. Thus, the possible roots that define the kernel of the FrFT or the resulting fractional wavelet cannot be obtained from a zero root.

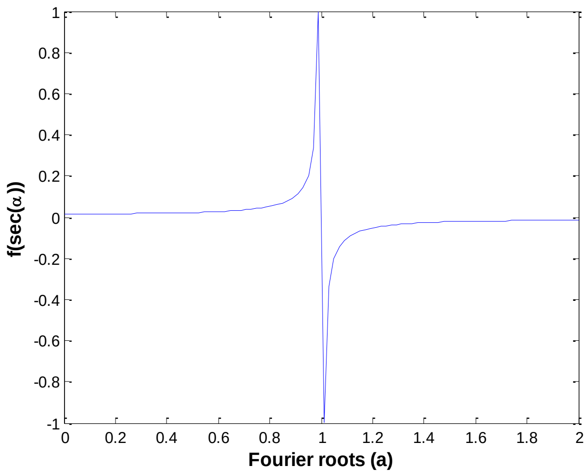

Fig. 1. A graph of the possible fractional Fourier order

From Equation 11 $\tan^{1/2}\alpha$ is the scale parameter. Thus, Equation 11 is a fractional Fourier transform in the resemblance of a wavelet transform. If the wavelet is of the quadratic phase function where $\psi(t)=\exp(i\pi t^2)$, as of Equation 1 where the coordinate is scaled by $\tan^{1/2}\alpha$ and the amplitude is scaled by A_α, then the discussion can proceed for signal of interest to exploit the FrFT property and wavelet property also. So, the convolution in the integral of Equation 11 is a wavelet transform. Notice that Equation 11 characterizes one–dimensional (1-D) wavelet transform only. However, for two-dimensional (2-D) and so on wavelet transforms, the following definitions must be followed [9]:

$$\psi(t,z)=e^{i\pi(t^2+z^2)} \tag{12}$$

Equation 12 is a typical 2-D wavelet transform. The scaled and dilated equivalent of Equation 12 can be expressed as:

$$\psi_{(k,\tau,\xi)}(t,z)=\exp\left[i\pi\left(\left(\frac{t-\tau}{k}\right)^2+\left(\frac{z-\xi}{k}\right)^2\right)\right] \tag{13}$$

where τ is the shift parameter respective to t-coordinate, ξ is the shift parameter respective to the z-coordinate. Recall the quadratic phase function based wavelet transform in Equation 9 and the fractional Fourier transform based wavelet constructed in Equation 11. We can define the phase function wavelet transformation based on Equation 11 by substituting the scaling factor k in Equation 6 into Equation 9 so that:

$$x(\tau,k)=\frac{1}{\tan^{1/2}\alpha}\int_{-\infty}^{+\infty}\exp\left[i\pi\left(\frac{t-\tau}{\tan^{1/2}\alpha}\right)^2\right]f(\tau)d\tau \tag{14}$$

where $k=\tan^{1/2}\alpha$. Thus in 1-D respect, if the function $f(\tau)$ is translated/dilated by τ then the t coordinate will be scaled by $\tan^{1/2}\alpha$. This gives a unique wavelet transform for every possible fractional Fourier order. This suggests that, instead of scaling $f(\tau)$ by some discrete factors (such as $k = 2^d$), the signal $f(\tau)$ can be scaled by the rotation factor of the fractional Fourier order ($k=\tan^{1/2}\alpha$). It can be further stressed that for every frequency content, the shift and fractional rotation (or frequency fractional) content is well localized. The properties of the new wavelet function can be studied based on the fractional Fourier transform wavelet (fractional-wave) discussed in [3] and extended in [2, 5]. It exploits the time-frequency MRA advantage of the WT and the fractional frequency domain explanation of an input signal advantage of the FrFT. Thus, the new transform can provide information on a signal of interest in a fractional scale during the MRA in the time-frequency plane.

IV. Closed-Form Expression of the Proposed Wavelet Transform

We can proceed to finding the exact discrete closed-form expression of Equation 14 while assuming that $k=\tan^{1/2}\alpha$ for brevity. Without loss of generality, recall the WT defined in Equation 9 can be expanded to accommodate the FrWT starting from:

$$\begin{aligned}
x(\tau,k)&=\frac{1}{k}\int_{-\infty}^{+\infty}\exp\left[i\pi\left(\frac{t}{k}-\frac{\tau}{k}\right)^2\right]f(\tau)d\tau\\
&=\frac{1}{k}\int_{-\infty}^{+\infty}\exp\left[i\pi\left\{\left(\frac{t}{k}\right)^2+\left(\frac{\tau}{k}\right)^2-\frac{2\tau t}{k^2}\right\}\right]f(\tau)d\tau\\
&=\frac{1}{k}\int_{-\infty}^{+\infty}\exp\left[i\pi\left(\frac{t}{k}\right)^2\right].\exp\left[i\pi\left\{\left(\frac{\tau}{k}\right)^2-\frac{2\tau t}{k^2}\right\}\right]f(\tau)d\tau\\
&=\frac{1}{k}\exp\left[\frac{i\pi t^2}{k^2}\right]\times\int_{-\infty}^{+\infty}\exp\left[i\pi\left\{\frac{\tau^2}{k^2}-\frac{2\tau t}{k^2}\right\}\right]f(\tau)d\tau
\end{aligned}$$

So,

$$x(\tau,k)=\frac{1}{k}\exp\left[\frac{i\pi t^2}{k^2}\right]\times\int_{-\infty}^{+\infty}\exp\left[\frac{i\pi}{k^2}\left\{\tau^2-2\tau t\right\}\right]f(\tau)d\tau$$

Let $y=\frac{i\pi}{k^2}\left(\tau^2-2\tau t\right)$ such that,

$$\begin{aligned}
x(\tau,\tan^{1/2}\alpha)&=\frac{e^{i\pi t^2/k^2}}{k}\int_{-\infty}^{+\infty}\exp[y]\,f(\tau)d\tau\\
&=\frac{e^{i\pi t^2/k^2}}{k}\int_{-\infty}^{+\infty}\exp[y]\,d\tau\, f(\tau)
\end{aligned} \tag{15}$$

But from traditional derivative theory knowledge and taking the first derivative of y,

$$\frac{dy}{d\tau}=\frac{i\pi}{k^2}(2\tau-2t)$$

$$= \frac{2i\pi}{k^2}(\tau - t) \tag{16a}$$

From Equation 16a, it can be rewritten that:

$$\Rightarrow d\tau = \frac{dy}{\left(\frac{2i\pi}{k^2}(\tau - t)\right)} \tag{16b}$$

Now, substituting for $d\tau$: Put Equation 16b into Equation 15,

$$x(\tau,k) = \frac{e^{i\pi t^2/k^2}}{k} \times \int_{-\infty}^{+\infty} \exp[y] \frac{dy}{\left(\frac{2i\pi}{k^2}(\tau - t)\right)} f(\tau)$$

$$= G \times \int_{-\infty}^{+\infty} \exp[y] \frac{dy}{\left(\frac{2i\pi}{k^2}(\tau - t)\right)} f(\tau)$$

where $G = \frac{e^{i\pi t^2/k^2}}{k}$, and remember that $k = \tan^{1/2}\alpha$:

$$x(\tau,k) = G \times \int_{-\infty}^{+\infty} \exp[y] \times \frac{1}{\left(\left(\frac{2i\pi}{k^2}\right)(\tau - t)\right)} \times dy\ f(\tau)$$

$$= \frac{1}{\left(\left(\frac{2i\pi}{k^2}\right)(\tau - t)\right)} \times G \times \int_{-\infty}^{+\infty} \exp[y]\, dy\ f(\tau)$$

$$= \frac{1}{\left(\left(\frac{2i\pi}{k^2}\right)(\tau - t)\right)} \times G \times \left[e^y\right]_{-\infty}^{+\infty} f(\tau)$$

The lower limit of the above relation is 0 (This is because $e^{-\infty} = 0$). However, recall that $y = \frac{i\pi}{k^2}\left(\tau^2 - 2\tau t\right)$, so

$$x(\tau,k) = \frac{1}{\left(\frac{2i\pi}{k^2}(\tau - t)\right)} \times G \times \exp\left[\frac{i\pi}{k^2}\left\{\infty^2 - 2t.\infty\right\}\right] f(\tau)$$

For discrete values of above expression, say $0 \le n < N$. Then,

$$x(n,k) = \sum_{n=0}^{N-1} \frac{1}{\left(\frac{2i\pi}{k^2}(n - t)\right)} \times G \times \exp\left[\frac{i\pi}{k^2}\left(n^2 - 2tn\right)\right] f(n) \tag{17}$$

By factoring the *n* terms accordingly,

$$x(n,k) = \sum_{n=0}^{N-1} \frac{1}{\left(\frac{2i\pi(n-t)}{k^2}\right)} \times G \times \exp\left[\frac{ni\pi}{k^2}(n - 2t)\right] f(n) \tag{18}$$

Substituting for *G* in Equation 18 with $k = \tan^{1/2}\alpha$,

$$x(n,k) = \sum_{n=0}^{N-1} \frac{(k)^2}{2i\pi(n-t)} \times \frac{e^{i\pi t^2/(k^2)}}{k} \times \exp\left[\frac{ni\pi}{k^2}\{n - 2t\}\right] f(n)$$

Eliminating the *k* terms appropriately, Equation 18 can be written in a compact form such that:

$$x(n,k) = \sum_{n=0}^{N-1} \frac{(k)}{2i\pi(n-t)} \cdot e^{i\pi t^2/(k)^2} \times \exp\left[\frac{ni\pi}{k^2}\{n - 2t\}\right] f(n)$$

On the other hand,

$$x(n,k) = k \sum_{n=0}^{N-1} \frac{e^{i\pi t^2/(k)^2}}{2i\pi(n-t)} \times \exp\left[\frac{ni\pi}{k^2}\{n - 2t\}\right] f(n)$$

Factorizing the terms accordingly,

$$x(n,k) = \frac{k\, e^{i\pi t^2/\tan\alpha}}{2i\pi} \sum_{n=0}^{N-1} \frac{1}{n-t} \times \exp\left[\frac{ni\pi}{\tan\alpha}\{n - 2t\}\right] f(n) \tag{19}$$

From Equation 19, the term $(k\, e^{i\pi t^2/\tan\alpha})/2i\pi$ can be seen as a normalization parameter in the discrete sense. Similarly to the formulation of Equation, from Equation 11, the discrete relation can be expressed as:

$$x(u,\tau) = \sum_{n=0}^{N-1} A_\alpha D_\alpha \frac{G_{new}}{\frac{2i\pi(n-u)}{k^2}} \times \exp\left[\frac{ni\pi}{k^2}\{n - 2u\}\right] f(n) \tag{20}$$

Or,

$$x(u,\tau) = A_\alpha D_\alpha \sum_{n=0}^{N-1} \frac{k^2 G_{new}}{2i\pi(n-u)} \times \exp\left[\frac{ni\pi}{k^2}\{n - 2u\}\right] f(n)$$

where,

$$G_{new} = \frac{e^{i\pi t^2/k^2}}{k} = \frac{e^{i\pi u^2/(k)^2}}{k}$$

So that,

$$x(u,\tau) = A_\alpha D_\alpha \sum_{n=0}^{N-1} \frac{k^2}{2i\pi(n-u)} \times \frac{e^{i\pi u^2/(k)^2}}{k} \times \exp\left[\frac{ni\pi}{k^2}\{n - 2u\}\right] f(n)$$

By eliminating the *k* terms appropriately, we obtain that:

$$x(u,\tau) = A_\alpha D_\alpha \sum_{n=0}^{N-1} \frac{k\, e^{i\pi u^2/(k)^2}}{2i\pi(n-u)} \times \exp\left[\frac{ni\pi}{k^2}\{n-2u\}\right] f(n) \qquad (21)$$

Now, recall that: $\hat{\alpha} = \operatorname{sgn}(\sin\alpha) = \frac{\sin\alpha}{|\sin\alpha|}$, then expanding A_α and D_α:

$$(AD)_\alpha = \frac{\exp[-i(\pi\hat{\alpha}/4 - \alpha/2)]}{\sqrt{\sin\alpha}} \times \exp(-i\pi u^2 \sin^2\alpha)$$

$$= \frac{\exp\left[-\left(i\pi\sin\alpha\left\{\frac{1}{4\times|\sin\alpha|} - u^2\sin\alpha\right\} - \frac{\alpha}{2}\right)\right]}{\sqrt{\sin\alpha}} \qquad (22)$$

Now, factorizing Equation 21:

$$x(u,\tau) = A_\alpha D_\alpha \frac{k\, e^{i\pi u^2/(k)^2}}{2i\pi} \sum_{n=0}^{N-1} \frac{1}{(n-u)} \times \exp\left[\frac{ni\pi}{k^2}\{n-2u\}\right] f(n) \qquad (23)$$

Substituting $(AD)_\alpha$ of Equation 22 for $A_\alpha D_\alpha$ in Equation 23:

$$x(u,\tau) = \frac{\exp\left[-\left(i\pi\sin\alpha\left\{\frac{1}{4\times|\sin\alpha|} - u^2\sin\alpha\right\} - \frac{\alpha}{2}\right)\right]}{\sqrt{\sin\alpha}} \times \frac{k\, e^{i\pi u^2/(k)^2}}{2i\pi}$$

$$\times \sum_{n=0}^{N-1} \frac{1}{(n-u)} \times \exp\left[\frac{ni\pi}{k^2}\{n-2u\}\right] f(n)$$

This can be written in a more compact form as:

$$x(u,\tau) = k \times \frac{\exp\left[-\left(\left\{\frac{i\pi\sin\alpha}{4\times|\sin\alpha|} - iu^2\sin^2\alpha\right\} - i\frac{\alpha}{2}\right) + \frac{i\pi u^2}{k^2}\right]}{2i\pi\sqrt{\sin\alpha}}$$

$$\times \sum_{n=0}^{N-1} \frac{1}{(n-u)} \times \exp\left[\frac{ni\pi}{k^2}\{n-2u\}\right] f(n)$$

But,

$$\frac{k}{\sqrt{\sin\alpha}} = \frac{\tan^{1/2}\alpha}{\sqrt{\sin\alpha}} = \frac{(\tan\alpha)^{1/2}}{\sqrt{\sin\alpha}} = \frac{\sqrt{\tan\alpha}}{\sqrt{\sin\alpha}} = \sqrt{\frac{1}{\cos\alpha}} = \frac{\sqrt{1}}{\sqrt{\cos\alpha}}$$

Clearly, Equation 23 can be exposed as:

$$x(u,\tau) = \frac{\exp\left[-\left(\left\{\frac{i\pi\sin\alpha}{4\times|\sin\alpha|} - iu^2\sin^2\alpha\right\} - i\frac{\alpha}{2}\right) + \frac{i\pi u^2}{k^2}\right]}{2i\pi\sqrt{\cos\alpha}}$$

$$\times \sum_{n=0}^{N-1} \frac{1}{(n-u)} \times \exp\left[\frac{ni\pi}{k^2}\{n-2u\}\right] f(n)$$

If $\gamma = \frac{\exp\left[-\left(\left\{\frac{i\pi\sin\alpha}{4\times|\sin\alpha|} - iu^2\sin^2\alpha\right\} - i\frac{\alpha}{2}\right) + \frac{i\pi u^2}{k^2}\right]}{2i\pi\sqrt{\cos\alpha}}$ is taken to be the normalization parameter for the alternative earlier fractional wavelet transform comparable to the proposed FrWT in Equation 19, then Equation 23 becomes:

$$x(u,\tau) = \gamma \times \sum_{n=0}^{N-1} \frac{1}{(n-u)} \times \exp\left[\frac{ni\pi}{\tan\alpha}\{n-2u\}\right] f(n) \qquad (24)$$

Now, comparing Equations 24 and 19, it can be observed that the complexity overhead associated with Equation 24 is more than that of Equation 19 (proposed). The computational cost of implementing FrWT based on Equation 11 is discouraging which is greatly reduced in the case of Equation 14. Meanwhile, that there are different wavelet for each fractional Fourier order, the idea of compact support stressed in [18] can be well accommodated. Also in the combination of the MRA property of the wavelet transform and the fractional Fourier property of the FrFT, the proposed well addresses a new wavelet that can achieve MRA in a fractional domain sense.

V. Conclusion

A new kernel for discussing the wavelet transform has been presented. It was derived from the fractional Fourier properties of the fractional Fourier transform to exploit the wavelet transform properties. The new wavelet combines the MRA and the fractional Fourier properties to discuss input signal in the fractional domain sense. Analytical results obtained were explicitly described in discrete and closed form solution unlike any work before. Also, it was identified that the fractional wavelet transform presented shows uniquely different wavelet for every particular fractional Fourier order. The new fractional wavelet obtained was shown to be computational efficient that the earlier fractional wavelet using explicit discrete definition shown.

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