Free transverse vibrations of a triple-beam system

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In this paper, the free vibration of an undamped triple-beam system on elastic foundation is analysed. The system consists of three Euler-Bernoulli beams of the same length that are arranged in parallel and continuously connected by elastic layers. The natural frequencies and the mode shapes of the system are determined. Furthermore, the initial value problem is solved to find the time-dependent free vibrations of the beams. Using a numerical example, the effects of the layer stiffnesses and those of the masses of the beams per unit length on the natural frequencies and the mode shapes are investigated in detail.

K e y w o r d s: vibration of beams, triple-beam systems, layer stiffness

1. Introduction

The vibration problem of beam systems is of particular importance in mechanical, industrial, aeronautical, and civil engineering. The vibration theory of single-beam systems is well developed and studied in detail in hundreds of contributions. However, there are only few contributions dealing with the vibrations of multi-beam systems. That is, probably, because of the difficulty in treating the governing coupled partial differential equations.

The free vibrations of two parallel simply supported beams continuously joined by a Winkler elastic layer are studied in [1]. Natural frequencies and mode shapes of vibration of the studied double-beam system are determined. Also the initial-value problem is considered to find the final form of free vibrations. An exact method for solving the vibration problem of a damped double-beam system subjected to harmonic excitation is presented in [2]. The handled double-beam system consists of two identical beams with the same boundary conditions on the same side. The beams are continuously connected by a viscoelastic layer. Natural frequencies and mode shapes of vibration of the system are determined and the forced vibrations

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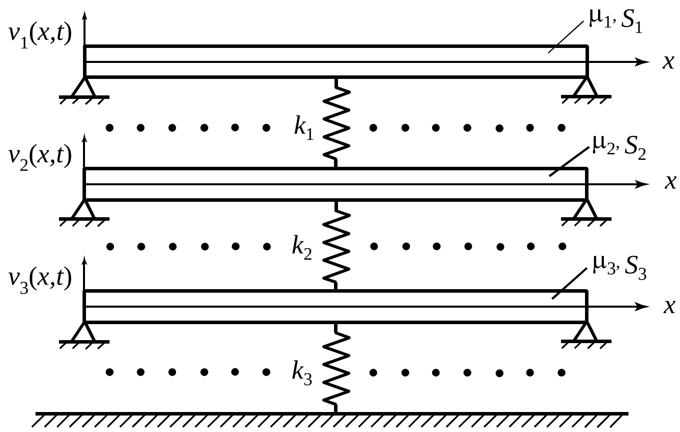

Fig. 1. Triple-beam system.

are investigated. Reference [3] treated the undamped forced transverse vibrations of an elastically connected complex simply supported double-beam system for several kinds of excitation loadings. The natural frequencies of a clamped-free double--beam system carrying tip masses to which several spring mass systems are attached across the span are determined in [4]. In [5], free vibrations of a double-beam system connected by many translational springs are studied. The bending vibrations of beams coupled by one and several double spring-mass systems are investigated in [6, 7]. Reference [8] presents a method of analysis of the free vibration of a three--layer sandwich beam with an elastic or viscoelastic core and arbitrary boundary conditions. The characteristic equation of the free vibration is derived by applying the Green function, which is obtained as a discrete solution of differential equations governing the flexural behaviour of the sandwich beam under the action of a concentrated load. In the paper, the natural frequencies of the studied beam are determined. Also the effect of the elastic or viscoelastic core shear modulus and the depth of the core on the natural frequency and the loss factor of the sandwich beam are evaluated.

In this paper, the free vibration behaviour of an undamped triple-beam system on elastic foundation as shown in Fig. 1 is analysed. The system consists of three beams of the same length that are arranged in parallel and continuously connected by elastic layers. The motion of the system is described by a set of homogenous coupled partial differential equations, which are solved by using the method of the assumed mode summation. The natural frequencies and the mode shapes of vibration of the beam system are determined and the initial value problem is solved to find the time-dependent free vibrations of the beams. Using a numerical example based on a simply supported triple-beam system, the effects of the layer stiffnesses and those of the masses of the beams per unit length on the natural frequencies and mode shapes are investigated in detail. Also it is proved that single and double--beam systems can be obtained from a triple-beam system by letting some of the system parameters go to zero or infinity.

2. Mathematical formulation

The system under consideration consists of three parallel, uniform, elastic, and homogeneous Euler-Bernoulli beams. The beams are joined by Winkler elastic layers and have the same length as shown in Fig. 1. The undamped transverse free vibrations of the system with general boundary conditions are described by the following coupled partial differential equations:

$$S_1 v_1'''' + k_1(v_1 - v_2) + \mu_1 \ddot{v}_1 = 0, \tag{1}$$

$$S_2 v_2'''' + k_1(v_2 - v_1) + k_2(v_2 - v_3) + \mu_2 \ddot{v}_2 = 0, \tag{2}$$

$$S_3 v_3'''' + k_2(v_3 - v_2) + k_3 v_3 + \mu_3 \ddot{v}_3 = 0, \tag{3}$$

where the constants S_i, μ_i, and k_i $(i = 1, 2, 3)$ are the flexural rigidity of the ith beam, the mass per unit length of the ith beam, and the stiffness modulus of the ith Winkler elastic layer, respectively. $v_i(x, t)$ denotes the deflection of the ith beam at position x and time t. A prime denotes differentiation with respect to spatial coordinate x and an overdot denotes differentiation with respect to time t.

The initial conditions are given as:

$$v_i(x, 0) = v_{i0}(x), \qquad \dot{v}_i(x, 0) = w_{i0}(x), \qquad i = 1, 2, 3. \tag{4}$$

In modal form, the beam deflection $v_i(x, t)$ is written as:

$$v_i(x, t) = \sum_{n=1}^{\infty} X_{in}(x) y_{in}(t), \tag{5}$$

where $X_{in}(x)$ is the known mode shape function for a single beam given as:

$$X_{in}(x) = \sin \kappa_{in} x + a_{in} \cos \kappa_{in} x + b_{in} \sinh \kappa_{in} x + c_{in} \cosh \kappa_{in} x, \tag{6}$$

where κ_{in} is the nth eigenfrequency parameter of the ith beam and $y_{in}(t)$ is the nth generalized deflection of the ith beam. Substituting the general solution (5) into Eqs. (1) to (3) and assuming that the three beams have the same length and the same boundary conditions ($X_{in} = X_n$) leads under consideration the relation $X_n'''' = \kappa_n^4 X_n$ to the following equations:

$$\sum_{n=1}^{\infty} \left[\mu_1 \ddot{y}_{1n} + k_{11} y_{1n} + k_{12} y_{2n}\right] X_n = 0, \tag{7}$$

$$\sum_{n=1}^{\infty} \left[\mu_2 \ddot{y}_{2n} + k_{21} y_{1n} + k_{22} y_{2n} + k_{23} y_{3n}\right] X_n = 0, \tag{8}$$

$$\sum_{n=1}^{\infty} \left[\mu_3 \ddot{y}_{3n} + k_{32} y_{2n} + k_{33} y_{3n}\right] X_n = 0, \tag{9}$$

where

$$\begin{aligned} k_{11} =& k_1 + \kappa_n^4 S_1, \qquad k_{22} = k_1 + k_2 + \kappa_n^4 S_2, \qquad k_{33} = k_2 + k_3 + \kappa_n^4 S_3, \\ k_{12} =& -k_1, \qquad k_{21} = -k_1, \qquad k_{23} = -k_2, \qquad k_{32} = -k_2. \end{aligned} \tag{10}$$

Multiplying Eqs. (7) to (9) by the eigenfunction $X_j(x)$ and integrating with respect to x between 0 and the beam's length L leads under consideration of the orthogonality conditions

$$\int_0^L \mu_i X_n X_j \mathrm{d}x = 0, \qquad n \neq j \tag{11}$$

to the system of homogeneous ordinary differential equations of the nth mode of the generalized deflections:

$$\mu_1 \ddot{y}_{1n} + k_{11} y_{1n} + k_{12} y_{2n} = 0, \tag{12}$$

$$\mu_2 \ddot{y}_{2n} + k_{21} y_{1n} + k_{22} y_{2n} + k_{23} y_{3n} = 0, \tag{13}$$

$$\mu_3 \ddot{y}_{3n} + k_{32} y_{2n} + k_{33} y_{3n} = 0. \tag{14}$$

From these equations it can be observed that the mass matrix for the nth mode is diagonal and the stiffness matrix is symmetric.

Assuming that each point of the three beams undergoes harmonic motion of the same frequency, we take solutions of Eqs. (12) to (14) as [9]:

$$y_{in}(t) = Y_{in} \cos \omega_n t, \qquad i = 1, 2, 3, \tag{15}$$

where ω_n is the nth natural frequency of the system and Y_{in} are constants that denote the amplitudes of y_{in}. Introducing Eq. (15) and the second time derivatives into Eqs. (12) to (14) leads to the following system of homogeneous algebraic equations:

$$\begin{bmatrix} k_{11}-\mu_1\omega_n^2 & k_{12} & 0 \\ k_{21} & k_{22}-\mu_2\omega_n^2 & k_{23} \\ 0 & k_{32} & k_{33}-\mu_3\omega_n^2 \end{bmatrix} \begin{bmatrix} Y_{1n} \\ Y_{2n} \\ Y_{3n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{16}$$

Equation (16) has nontrivial solution of Y_{1n}, Y_{2n} and Y_{3n}, when the determinant of the coefficient matrix is equal to zero. This leads to the following frequency equation:

$$\begin{aligned} \mu_1\mu_2\mu_3\omega_n^6 &- (k_{11}\mu_2\mu_3 + k_{22}\mu_1\mu_3 + k_{33}\mu_1\mu_2)\omega_n^4 \\ &- (k_{12}k_{21}\mu_3 + k_{23}k_{32}\mu_1 - k_{11}k_{22}\mu_3 - k_{11}k_{33}\mu_2 - k_{22}k_{33}\mu_1)\omega_n^2 \\ &- k_{11}k_{22}k_{33} + k_{11}k_{23}k_{32} + k_{12}k_{21}k_{33} = 0. \end{aligned} \tag{17}$$

Solving this equation yields a set of three natural frequencies ω_{sn}, $s = 1, 2, 3$, of the system for each eigenfrequency parameter κ_n of a single beam.

Now solutions (15) may be written as:

$$y_{1n}(t) = \sum_{s=1}^{3} T_{sn}(t), \tag{18}$$

$$y_{2n}(t) = \sum_{s=1}^{3} r_{2sn} T_{sn}(t), \tag{19}$$

$$y_{3n}(t) = \sum_{s=1}^{3} r_{3sn} T_{sn}(t), \tag{20}$$

where

$$T_{sn}(t) = A_{sn}\cos\omega_{sn}t + B_{sn}\sin\omega_{sn}t, \qquad s = 1, 2, 3. \tag{21}$$

The mode shape coefficients r_{2sn} and r_{3sn} are obtained from Eq. (16) and are given as:

$$r_{2sn} = \frac{\mu_1 \omega_{sn}^2 - k_{11}}{k_{12}}, \tag{22}$$

$$r_{3sn} = \frac{k_{32}}{k_{12}} \frac{\mu_1 \omega_{sn}^2 - k_{11}}{\mu_3 \omega_{sn}^2 - k_{33}}. \tag{23}$$

The mode coefficient r_{isn} has three subscripts: the first one i denotes the corresponding beam for which r_{isn} is calculated ($i = 2, 3$), the second subscript s denotes the sth natural frequency of the system ($s = 1, 2, 3$), and the third subscript denotes the nth eigenfrequency parameter κ_n of a single beam. Substituting Eqs. (18) to (20) into Eq. (5) yields for the free transverse vibration of the triple-beam system the following equations:

$$v_1(x,t) = \sum_{n=1}^{\infty} X_n(x) y_{1n}(t) = \sum_{n=1}^{\infty} \sum_{s=1}^{3} X_{1sn}(x) T_{sn}(t), \tag{24}$$

$$v_2(x,t) = \sum_{n=1}^{\infty} X_n(x) y_{2n}(t) = \sum_{n=1}^{\infty} \sum_{s=1}^{3} X_{2sn}(x) T_{sn}(t), \tag{25}$$

$$v_3(x,t) = \sum_{n=1}^{\infty} X_n(x) y_{3n}(t) = \sum_{n=1}^{\infty} \sum_{s=1}^{3} X_{3sn}(x) T_{sn}(t), \tag{26}$$

where

$$X_{1sn}(x) = X_n(x), \tag{27}$$

$$X_{2sn}(x) = r_{2sn} X_n(x), \tag{28}$$

$$X_{3sn}(x) = r_{3sn} X_n(x) \tag{29}$$

are the normal mode shapes of vibration of the triple beam system corresponding to the natural frequencies ω_{sn}.

The unknown constants A_{sn} and B_{sn} in the time functions T_{sn} are obtained from the initial conditions given in Eq. (4). Substituting Eqs. (24) through (26) and their first time derivatives into Eq. (4) leads to the following equations:

$$v_{10}(x) = \sum_{n=1}^{\infty} X_n [A_{1n} + A_{2n} + A_{3n}], \tag{30}$$

$$v_{20}(x) = \sum_{n=1}^{\infty} X_n[r_{21n}A_{1n} + r_{22n}A_{2n} + r_{23n}A_{3n}], \tag{31}$$

$$v_{30}(x) = \sum_{n=1}^{\infty} X_n[r_{31n}A_{1n} + r_{32n}A_{2n} + r_{33n}A_{3n}], \tag{32}$$

$$w_{10}(x) = \sum_{n=1}^{\infty} X_n[\omega_{1n}B_{1n} + \omega_{2n}B_{2n} + \omega_{3n}B_{3n}], \tag{33}$$

$$w_{20}(x) = \sum_{n=1}^{\infty} X_n[r_{21n}\omega_{1n}B_{1n} + r_{22n}\omega_{2n}B_{2n} + r_{23n}\omega_{3n}B_{3n}], \tag{34}$$

$$w_{30}(x) = \sum_{n=1}^{\infty} X_n[r_{31n}\omega_{1n}B_{1n} + r_{32n}\omega_{2n}B_{2n} + r_{33n}\omega_{3n}B_{3n}]. \tag{35}$$

Multiplying Eqs. (30) to (35) by the eigenfunction X_j and then integrating with respect to x from 0 to L and considering the orthogonality conditions:

$$\int_0^L \mu_i X_j X_n \mathrm{d}x = m_n \delta_{nj}, \tag{36}$$

where δ_{nj} is the Kronecker delta and m_n is the nth generalized mass of a single beam, leads to the following equations:

$$A_{1n} + A_{2n} + A_{3n} = \frac{1}{e_n}\int_0^L v_{10}X_n\mathrm{d}x, \tag{37}$$

$$r_{21n}A_{1n} + r_{22n}A_{2n} + r_{23n}A_{3n} = \frac{1}{e_n}\int_0^L v_{20}X_n\mathrm{d}x, \tag{38}$$

$$r_{31n}A_{1n} + r_{32n}A_{2n} + r_{33n}A_{3n} = \frac{1}{e_n}\int_0^L v_{30}X_n\mathrm{d}x, \tag{39}$$

$$\omega_{1n}B_{1n} + \omega_{2n}B_{2n} + \omega_{3n}B_{3n} = \frac{1}{e_n}\int_0^L w_{10}X_n\mathrm{d}x, \tag{40}$$

$$r_{21n}\omega_{1n}B_{1n} + r_{22n}\omega_{2n}B_{2n} + r_{23n}\omega_{3n}B_{3n} = \frac{1}{e_n}\int_0^L w_{20}X_n\mathrm{d}x, \tag{41}$$

$$r_{31n}\omega_{1n}B_{1n} + r_{32n}\omega_{2n}B_{2n} + r_{33n}\omega_{3n}B_{3n} = \frac{1}{e_n}\int_0^L w_{30}X_n\mathrm{d}x, \tag{42}$$

where $e_n = m_n/\mu$. Solving Eqs. (37) through (42) yields the unknown constants:

$$A_{1n} = \frac{1}{r_n}\int_0^L [(r_{23n}r_{32n} - r_{22n}r_{33n})v_{10} + (r_{33n} - r_{32n})v_{20} + (r_{22n} - r_{23n})v_{30}]\, X_n \mathrm{d}x, \tag{43}$$

$$A_{2n} = \frac{1}{r_n}\int_0^L [(r_{21n}r_{33n} - r_{23n}r_{31n})v_{10} + (r_{31n} - r_{33n})v_{20} + (r_{23n} - r_{21n})v_{30}]\, X_n \mathrm{d}x, \tag{44}$$

$$A_{3n} = \frac{1}{r_n}\int_0^L [(r_{22n}r_{31n} - r_{21n}r_{32n})v_{10} + (r_{32n} - r_{31n})v_{20} + (r_{21n} - r_{22n})v_{30}]\, X_n \mathrm{d}x, \tag{45}$$

$$B_{1n} = \frac{1}{r_n\omega_{1n}}\int_0^L [(r_{23n}r_{32n} - r_{22n}r_{33n})w_{10} + (r_{33n} - r_{32n})w_{20} + (r_{22n} - r_{23n})w_{30}]\, X_n \mathrm{d}x, \tag{46}$$

$$B_{2n} = \frac{1}{r_n\omega_{2n}}\int_0^L [(r_{21n}r_{33n} - r_{23n}r_{31n})w_{10} + (r_{31n} - r_{33n})w_{20} + (r_{23n} - r_{21n})w_{30}]\, X_n \mathrm{d}x, \tag{47}$$

$$B_{3n} = \frac{1}{r_n\omega_{3n}}\int_0^L [(r_{22n}r_{31n} - r_{21n}r_{32n})w_{10} + (r_{32n} - r_{31n})w_{20} + (r_{21n} - r_{22n})w_{30}]\, X_n \mathrm{d}x, \tag{48}$$

where

$$r_n = e_n[r_{21n}(r_{33n} - r_{32n}) + r_{22n}(r_{31n} - r_{33n}) + r_{23n}(r_{32n} - r_{31n})]. \tag{49}$$

3. Numerical evaluations

To show the effects of the layer stiffnesses k_i and the masses per unit length of the beams μ_i on the natural frequencies and mode shapes of the studied triple--beam system, a numerical example based on a simply supported triple-beam system ($X_{in}(0) = X_{in}(L) = 0$) is considered. The used initial values of parameters are as follows: $S_{i0} = 10$ MN·m^2, $k_{i0} = 0.2$ MN/m^2, $\mu_{i0} = 100$ kg/m, $(i = 1, 2, 3)$, $L = 10$ m.

Now, in order to explore the influence of each of the six parameters k_i, μ_i ($i = 1, 2, 3$) separately on the natural frequencies and/or mode shapes of the considered system, one of these parameters will be varied between zero and infinity while the other parameters remain constant. Then the natural frequencies ω_{sn} and mode shape coefficients r_{isn} will be calculated and given in form of tables or figures.

3.1 Effect of the parameters on the natural frequencies

Table 1 contains circular natural frequencies ω_{sn} of the studied triple-beam system by varying the layer stiffness k_1. Letting the layer stiffnesses k_i go to zero or to infinity leads to the simpler beam systems shown in Fig. 3. These systems consist of a combination of the single and double-beam systems shown in Fig. 2 with some zero elements. The natural frequencies ω_{sn} of the systems shown in Fig. 2 may be given as:

$$\omega_n = \kappa_n^2 \sqrt{\frac{S}{\mu} + \frac{k_1 + k_2}{\mu \kappa_n^4}} \tag{50}$$

for the single-beam system and

$$\omega_{1,2n} = \left[\frac{1}{2\mu_1\mu_2} \left[K_{1n}\mu_2 + K_{2n}\mu_1 \mp \sqrt{(K_{1n}\mu_2 + K_{2n}\mu_1)^2 - 4\mu_1\mu_2(K_{1n}K_{2n} - k_2^2)} \right] \right]^{-1/2} \tag{51}$$

for the double-beam system (derivation of Eq. (51) is given in the Appendix), where

$$K_{1n} = k_1 + k_2 + \kappa_n^4 S_1, \tag{52}$$

Table 1. Circular natural frequencies of a triple-beam system ω_{sn} by varying the values of the layer stiffness $k_1 = \alpha k_{10}$

	$n = 1$			$n = 2$			$n = 3$		
α	ω_{11}	ω_{21}	ω_{31}	ω_{12}	ω_{22}	ω_{32}	ω_{13}	ω_{23}	ω_{33}
0	31.21	41.69	78.81	124.84	127.87	144.30	280.89	282.25	290.07
0.1	33.68	43.80	79.18	125.48	128.57	144.50	281.18	282.57	290.17
0.2	35.02	46.45	79.61	125.85	129.50	144.74	281.34	282.99	290.29
0.5	36.45	54.53	81.36	126.25	132.61	145.71	281.52	284.43	290.77
1	37.02	63.91	86.42	126.42	136.73	148.59	281.60	286.38	292.22
2	37.30	70.53	102.75	126.50	139.95	158.64	281.64	287.93	297.46
5	37.47	73.25	148.84	126.55	141.34	191.74	281.66	288.61	316.35
100	37.58	74.35	634.02	126.58	141.92	645.44	281.67	288.89	692.75
∞	37.58	74.40	–	126.59	141.94	–	281.67	288.90	–

Table 2. Circular natural frequencies of a triple-beam system ω_{sn} by varying the values of the mass per unit length of the first beam, $\mu_1 = \alpha\mu_{10}$

	$n=1$			$n=2$			$n=3$		
α	ω_{11}	ω_{21}	ω_{31}	ω_{12}	ω_{22}	ω_{32}	ω_{13}	ω_{23}	ω_{33}
0	46.81	80.07	–	132.17	146.54	–	284.39	291.34	–
0.1	46.08	79.34	176.84	132.12	146.49	419.66	284.38	291.33	899.48
0.2	45.25	78.32	128.99	132.06	146.43	297.02	284.38	291.33	636.06
0.5	42.33	72.76	93.87	131.71	146.01	188.88	284.34	291.29	402.37
1	37.02	63.91	86.42	126.42	136.73	148.59	281.60	286.38	292.22
2	29.20	58.52	84.61	92.76	133.02	147.19	201.01	284.47	291.42
5	19.49	55.91	83.89	58.88	132.72	147.00	127.16	284.44	291.39
100	4.48	54.60	83.53	13.18	132.62	146.92	28.43	284.43	291.38
∞	–	54.54	83.51	–	132.61	146.92	–	284.43	291.38

$$K_{2n} = k_2 + k_3 + \kappa_n^4 S_2, \qquad (53)$$

and μ and S are the mass per unit length and the flexural rigidity of a beam, respectively.

The natural frequencies ω_{sn} given in the first and last rows of Table 1 are equal to the frequencies for the systems shown in Figs. 3a and 3b, respectively. These frequencies are calculated using Eqs. (50) and (51).

It is evident from Table 1 that increasing the values of k_1 leads to increasing the natural frequencies ω_{sn} of the system. The increase in the natural frequencies is higher for higher values of s and smaller for higher values of n. The first ω_{1n} and second ω_{2n} natural frequencies vary in a limited region bounded with frequencies obtained from the systems given in Fig. 3. The third natural frequency ω_{3n} may have very high values by increasing the values of k_1 and goes to infinity when k_1

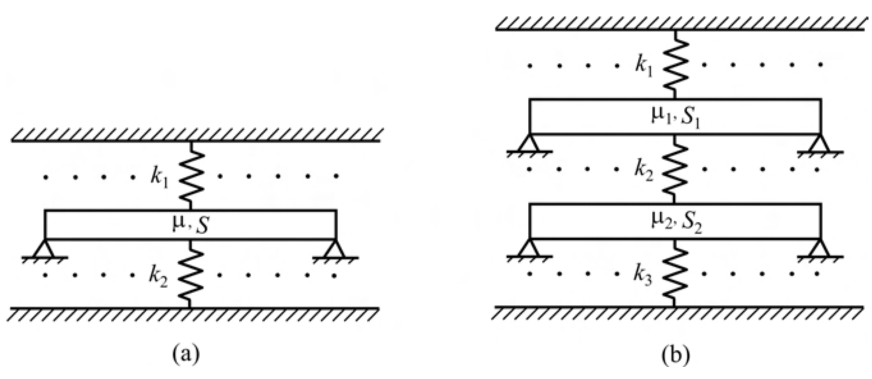

Fig. 2. (a) single-beam system, (b) double-beam system.

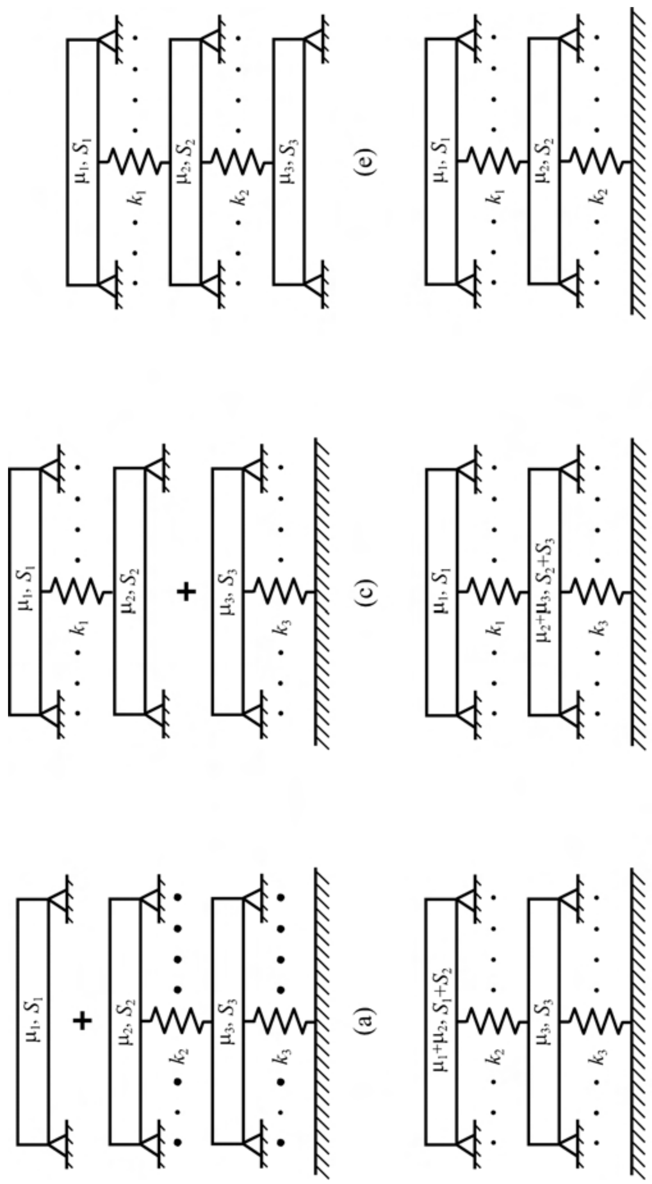

Fig. 3. Simple systems of a triple-beam system: (a) $k_1 \to 0$, (b) $k_1 \to \infty$, (c) $k_2 \to 0$, (d) $k_2 \to \infty$, (e) $k_3 \to 0$, (f) $k_3 \to \infty$.

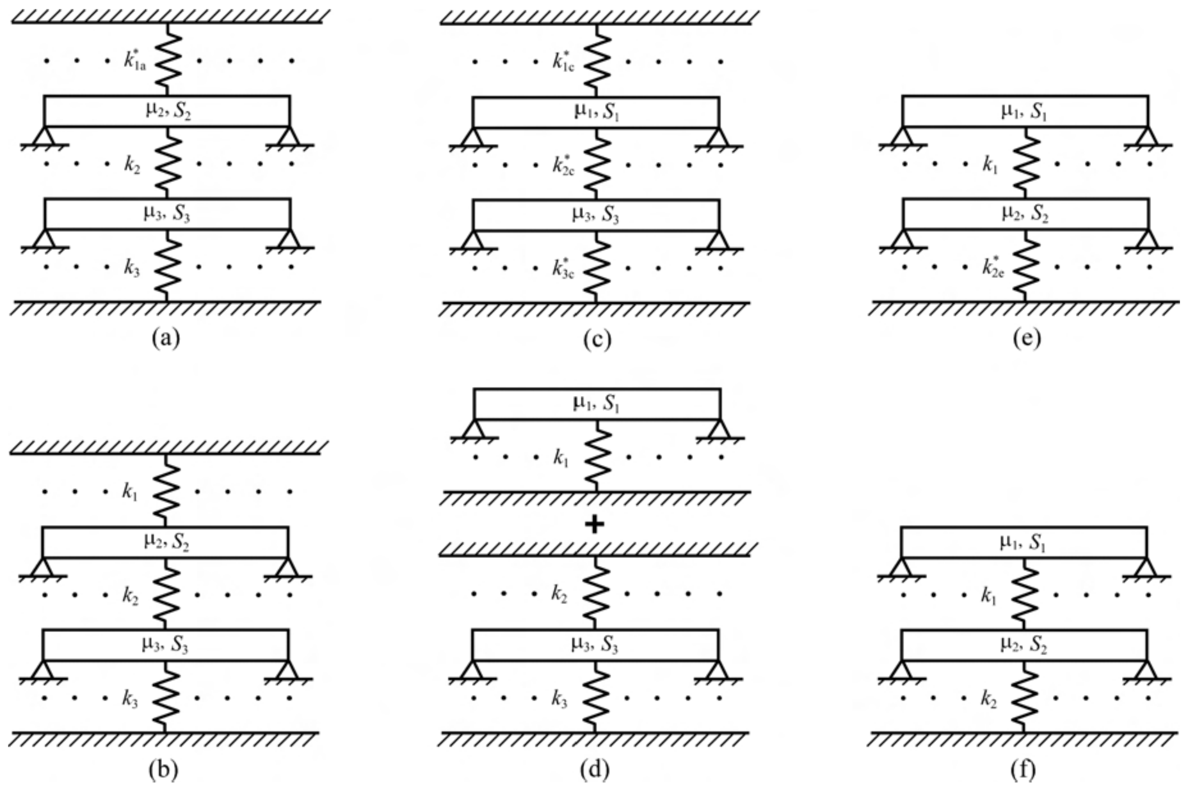

Fig. 4. Simple systems of a triple-beam system: (a) $\mu_1 \to 0$, (b) $\mu_1 \to \infty$, (c) $\mu_2 \to 0$, (d) $\mu_2 \to \infty$, (e) $\mu_3 \to 0$, (f) $\mu_3 \to \infty$. k_{1a}^*, k_{1c}^*, k_{2c}^*, k_{3c}^*, and k_{2e}^* are given in the Appendix.

approaches infinity. For the used numerical values of parameters, the first natural frequency ω_{1n} varies in frequency intervals with widths of 6.37, 1.75, and 0.78 rad/s for $n = 1, 2$, and 3, respectively, whereas the second natural frequency ω_{2n} varies in frequency intervals with widths of 32.71, 14.02, and 6.65 rad/s for $n = 1, 2$, and 3, respectively. Also the natural frequencies ω_{1n} in the first row of Table 1 are the same as those of a single beam simply supported. Varying k_2 and k_3 has, in general, a similar effect on the natural frequencies of the system as varying k_1.

Table 2 gives natural frequencies ω_{sn} of the triple beam system by varying the mass per unit length of the beam μ_1. Letting μ_i go to zero or to infinity leads to the simpler beam systems shown in Fig. 4. The modified stiffnesses k^*_{1a}, k^*_{1c}, k^*_{2c}, k^*_{2e}, and k^*_{3c}, which appear in Fig. 4, are given in the Appendix. From the Table 2 it is evident that increasing the values of μ_1 leads to decreasing the natural frequencies ω_{sn}. The natural frequencies ω_{sn} given in the first and last rows of this table are corresponding to the simpler systems shown in Figs. 4a and 4b, respectively. From the table, the following is observable:

The first natural frequency ω_{1n} (for each $n = 1, 2, 3$) has a limited value by very low values of μ_1. However, increasing the values of μ_1 leads to a decrease in this frequency. In the limiting case $\mu_1 \to \infty$, this frequency goes to zero. The third natural frequency ω_{3n} has very high values for very small values of μ_1 and becomes smaller when the values of μ_1 are increased. Also this frequency takes limited values as $\mu_1 \to \infty$. The second natural frequency ω_{2n} varies between two limited values. Varying μ_2 and μ_3 has, in general, a similar effect on the natural frequencies of the system as varying μ_1.

3.2 Effect of the parameters on the mode shapes

Figure 5 shows the mode coefficients r_{is1} versus the dimensionless coefficient $\alpha = k_i/k_{i0}$ by varying the layer stiffnesses k_1 through k_3. In the Fig. 5, the coefficients r_{is1} obtained by varying the layer stiffness k_i are denoted by k_i (i.e. r_{is1k_i}). From the figure, the following characteristics are noticeable:

1. The mode coefficients r_{211} and r_{311} are always positive regardless the values of k_i.

2. The mode coefficient r_{211} increases by increasing the values of k_1 but decreases by increasing the values of k_2 or k_3.

3. The magnitude (absolute value) of r_{221} increases continuously by increasing the values of k_2 or k_3. On the other hand, by increasing the values of k_1, the magnitude of r_{221} decreases down to zero for $\alpha = 2$ and then increases and goes asymptotically to 1 for $k_1 \to \infty$. Also r_{221} may have positive or negative values depending on the values of k_i.

4. The magnitude of the mode coefficient r_{321} decreases continuously by increasing the values of k_2. By increasing the values of k_1, the magnitude of r_{321} decreases to a minimum at $\alpha \approx 0.5$ and then increases. On the other hand, by increasing the values of k_3, the magnitude of r_{321} increases to a maximum at $\alpha \approx 0.9$

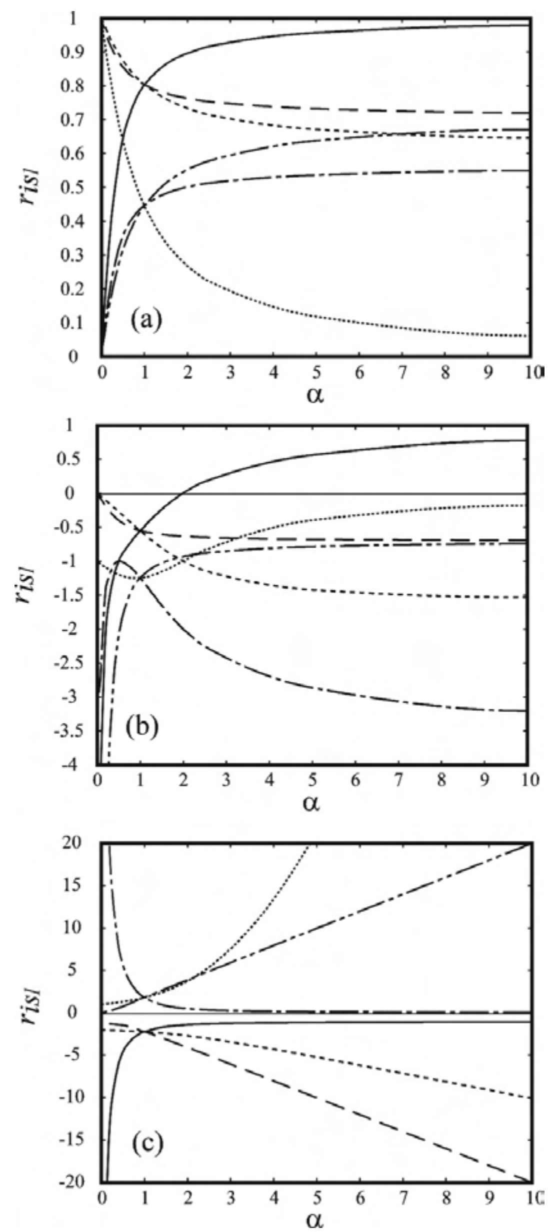

Fig. 5. Mode shape coefficients r_{is1} by varying the layer stiffnesses k_i for the first eigenfrequency parameter ($\kappa_1 = \pi/L$). (a) first mode ($s = 1$), (b) second mode ($s = 2$), (c) third mode ($s = 3$) of the triple-beam system. (———) r_{2s1k_1}, (– – –) r_{2s1k_2}, (- - - -) r_{2s1k_3}, (–·–·–) r_{3s1k_1}, (–··–··–) r_{3s1k_2}, (·········) r_{3s1k_3}.

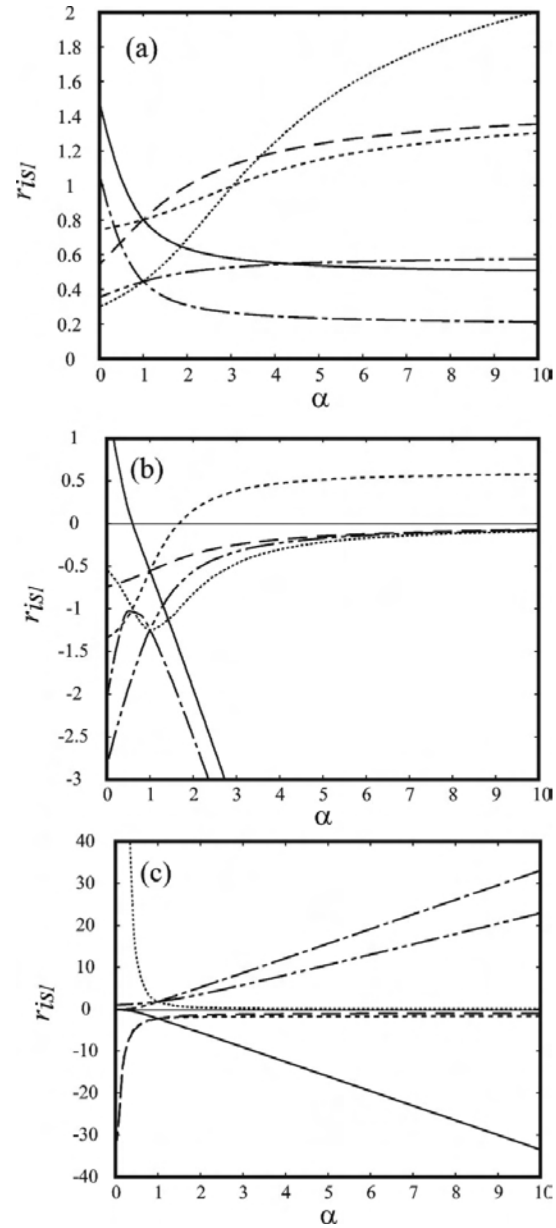

Fig. 6. Mode shape coefficients r_{is1} by varying the masses per unit length of the beams μ_i for the first eigenfrequency parameter ($\kappa_1 = \pi/L$). (a) first mode ($s = 1$), (b) second mode ($s = 2$), (c) third mode ($s = 3$) of the triple-beam system. (——) $r_{2s1\mu_1}$, (– – –) $r_{2s1\mu_2}$, (- - - -) $r_{2s1\mu_3}$, (–·–·–) $r_{3s1\mu_1}$, (–··–··–) $r_{3s1\mu_2}$, (·········) $r_{3s1\mu_3}$.

and then decreases and goes to zero for $k_3 \to \infty$. The coefficient r_{321} is always negative.

5. The magnitudes of mode coefficients r_{231} and r_{331} decrease by increasing the values of k_1 whereas these magnitudes increase by increasing the values of k_2 or k_3. Note that r_{231} is always negative while r_{331} is always positive, regardless the values of k_i.

6. From Fig. 5 it can be observed that all the curves of r_{2s1} go for $\alpha = 1$ through one point regardless the values of k_i. The same do the curves of r_{3s1}.

Figure 6 shows the mode coefficients r_{is1} versus the dimensionless coefficient $\alpha = \mu_i/\mu_{i0}$ by varying the masses of the beams per unit length μ_i for the first eigenfrequency parameter of a simply supported beam ($\kappa_1 = \pi/L$). In the Fig. 6, the coefficients r_{is1} obtained by varying the masses of the beams per unit length μ_i are denoted by μ_i (i.e. $r_{is1\mu_i}$). From the figure, the following characteristics are observed:

1. Figure 6a shows that the mode coefficients r_{211} and r_{311} decrease by increasing the values of μ_1 but increase by increasing the values of μ_2 or μ_3. Also r_{211} and r_{311} are always positive regardless the values of μ_i.

2. Figure 6b shows that by increasing the values of μ_1 or μ_3, the magnitude of the mode coefficient r_{221} decreases down to zero and then increases. Also the magnitude of r_{221} decreases continuously by increasing μ_2 and goes to zero as $\mu_2 \to \infty$. Note that depending on the values of μ_i, the coefficient r_{221} may take positive or negative values.

Table 3. Mode shape coefficients r_{isn} for $\alpha = 1$ and $\alpha = 2.5$

n	r_{isn}	$\alpha = 1$	$\alpha = 2.5$					
			k_1	k_2	k_3	μ_1	μ_2	μ_3
1	r_{211}	0.802	0.916	0.752	0.713	0.601	1.063	0.944
	r_{311}	0.445	0.512	0.578	0.222	0.282	0.515	0.836
	r_{221}	−0.555	0.170	−0.652	−1.141	−2.657	−0.295	0.298
	r_{321}	−1.247	−2.258	−0.882	−0.839	−3.204	−0.419	−0.613
	r_{231}	−2.250	−1.286	−5.100	−3.072	−7.405	−1.275	−1.750
	r_{331}	1.802	0.346	4.904	5.367	6.922	4.634	0.312
2	r_{212}	0.802	0.916	0.752	0.713	0.162	5.022	4.945
	r_{312}	0.445	0.512	0.578	0.222	0.026	0.834	28.401
	r_{222}	−0.555	0.170	−0.652	−1.141	−13.281	−0.074	0.593
	r_{322}	−1.247	−2.258	−0.882	−0.839	−13.791	−0.080	−0.055
	r_{232}	−2.250	−1.286	−5.100	−3.072	−18.259	−1.072	−1.663
	r_{332}	1.802	0.346	4.904	5.367	17.765	14.935	0.102
3	r_{213}	0.802	0.916	0.752	0.713	0.040	23.903	23.887
	r_{313}	0.445	0.512	0.578	0.222	0.002	0.960	593.452
	r_{223}	−0.555	0.170	−0.652	−1.141	−60.697	−0.016	0.613
	r_{323}	−1.247	−2.258	−0.882	−0.839	−61.199	−0.017	−0.011
	r_{233}	−2.250	−1.286	−5.100	−3.072	−65.695	−1.016	−1.629
	r_{333}	1.802	0.346	4.904	5.367	65.197	62.233	0.026

3. From Fig. 6b it is observable that the mode coefficient r_{321} is always negative regardless the values of μ_i. By increasing the values of μ_1, the magnitude of r_{321} decreases to a minimum at $\alpha \approx 0.59792$ before it increases continuously and becomes ∞ as $\mu_1 \to \infty$. On the other hand by increasing the values of μ_3, the magnitude of r_{321} increases to a maximum at $\alpha \approx 1.05$ and then decreases continuously and becomes zero for $\mu_3 \to \infty$. Also the magnitude of r_{321} decreases continuously by increasing the values of μ_2 and becomes zero as $\mu_2 \to \infty$.

4. From Fig. 6c it is evident that r_{231} is always negative whereas r_{331} is always positive regardless the values of μ_i. The magnitude of r_{231} increases by increasing μ_1 but decreases by increasing μ_2 or μ_3. On the other hand, the magnitude of r_{331} increases by increasing μ_1 or μ_2 but decreases by increasing the values of μ_3.

5. From Fig. 6 it can be observed that all the curves of r_{2s1} go for $\alpha = 1$ through one point regardless the values of μ_i. The same do the curves of r_{3s1}.

Table 3 gives mode shape coefficients r_{isn} for $\alpha = 1$ and $\alpha = 2.5$ for the first three eigenfrequency parameters $\kappa_n = n\pi/L$. The third column of the table contains the mode coefficients r_{isn} for $\alpha = 1$, i.e. for the initial values of parameters k_{i0}, μ_{i0}, and S_{i0}. The fourth column contains the mode coefficients r_{isn} for $k_1 = 2.5k_{10}$, where all other parameters retain their initial values. The contents of the following columns are to be similarly explained. From the Table 3, it is essentially noticeable that for a given value of the layer stiffnesses k_i, the mode coefficients r_{isn} are independent of the eigenfrequency parameter κ_n. However, different value of κ_n leads to different values of r_{isn} for given values of the masses per unit length of the beams μ_i. Furthermore, it is noticeable from the Table 3 that depending on the values of the system parameters, the mode coefficients r_{22n} may take positive or negative values.

Figure 7 shows mode shape functions X_{isn} of vibration of the studied triple--beam system corresponding to the first three sets of natural frequencies ω_{sn} for $\alpha = 1$. For other values of α, similar mode shapes will be obtained, but with different magnitudes. In general, a triple-beam system executes three kinds of vibrating motions, i.e., the system has three mode shapes of vibration for each eigenfrequency parameter κ_n. These mode shapes are given in a row in Fig. 7 for each κ_n, where the first, second, and third columns of the figure show the first, second, and third mode of vibration, respectively.

From the previous analysis it was found that the mode shape coefficients r_{21n} and r_{31n} corresponding to the first mode shape are always positive. As a result, in the first normal mode, all three beams move in phase. Furthermore it was found that depending on the values of the system parameters, the mode coefficient r_{22n} may be positive or negative whereas the mode shape coefficient r_{32n} is always negative. Noting that these coefficients are related to the second mode, the system can execute the following motions: either the second and third beams move in phase with each other and out of phase with the first beam or the first and second beams move in phase with each other and out of phase with the third beam. On the

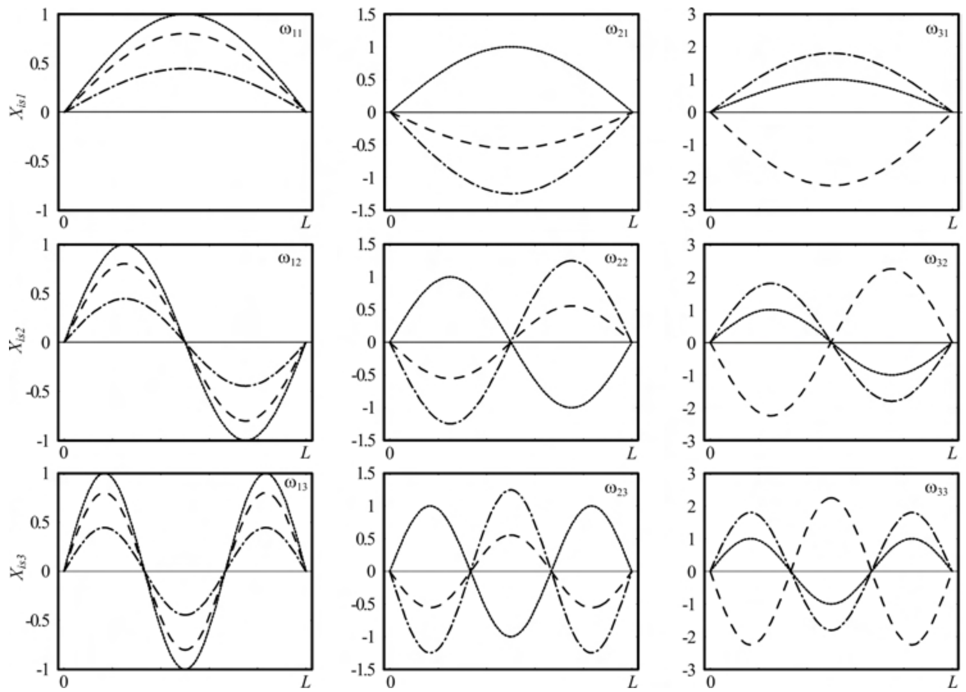

Fig. 7. Mode shapes of vibration of a triple-beam system corresponding to the first three sets of natural frequencies for the data: $S_{i0} = 10$ MN·m^2, $k_{i0} = 0.2$ MN/m^2, $\mu_{i0} = 100$ kg/m, $(i = 1, 2, 3)$, $L = 10$. (——) X_{1sn}, (– – –) X_{2sn}, (–·–·–) X_{3sn}.

other hand, it was found that the mode coefficient r_{23n} is always negative whereas the mode coefficient r_{33n} is always positive. These coefficients are related to the third mode. Consequently, in the third normal mode the first and third beams move in phase with each other and out of phase with the second beam.

4. Conclusions

The free vibrational behaviour of an undamped triple-beam system is analysed. The natural frequencies and mode shapes of the system are determined and discussed in detail. Also the effects of the layer stiffnesses and the masses per unit length of the beams on the natural frequencies and mode shapes of the system are explored. It is found that increasing the layer stiffnesses k_i leads to an increase in the natural frequencies, whereas increasing the masses per unit length of the beams μ_i leads to a decrease in the natural frequencies. Furthermore, it was observed that for a given κ_n the first natural frequency ω_{1n} varies in a frequency interval between zero and a limited value, the second frequency ω_{2n} varies between two limited values, and the third natural frequency ω_{3n} varies between a limited value and infinity. Also the natural frequencies are located closer to each other for higher values of κ_n.

In general, a triple-beam system executes three kinds of vibrating motions. From the analysis performed, it was obtained that the mode coefficients r_{21n}, r_{31n}, and r_{33n} are always positive, the mode coefficient r_{22n} may take positive or negative values, and the mode coefficients r_{23n} and r_{32n} are always negative. Note that the coefficients r_{21n} and r_{31n} refer to the first mode, the coefficients r_{22n} and r_{32n} refer to the second mode, and the coefficients r_{23n} and r_{33n} are related to the third mode.

Furthermore, it was found that single and double-beam systems can be considered as special cases of a triple-beam system by letting some of the system parameters go to zero or infinity.

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Appendix

$$k_{1a}^* = \frac{k_1 \kappa_n^4 S_1}{k_1 + \kappa_n^4 S_1}, \qquad k_{1c}^* = \frac{k_1 \kappa_n^4 S_2}{k_1 + k_2 + \kappa_n^4 S_2},$$

$$k_{2c}^* = \frac{k_1 k_2}{k_1 + k_2 + \kappa_n^4 S_2}, \qquad k_{3c}^* = \frac{k_2 \kappa_n^4 S_2}{k_1 + k_2 + \kappa_n^4 S_2} + k_3,$$

$$k_{2e}^* = \frac{k_2 (k_3 + \kappa_n^4 S_3)}{k_2 + k_3 + \kappa_n^4 S_3}.$$

Derivation of Eq. (51)

The undamped transverse free vibrations of the system shown in Fig. 2b are described by

$$S_1 v_1'''' + k_1 v_1 + k_2(v_1 - v_2) + \mu_1 \ddot{v}_1 = 0, \tag{A-1}$$

$$S_2 v_2'''' + k_2(v_2 - v_1) + k_3 v_2 + \mu_2 \ddot{v}_2 = 0. \tag{A-2}$$

Substituting Eq. (5) into Eqs. (A-1) and (A-2) and assuming that both beams have the same length and the same boundary conditions leads under consideration the relation $X_n'''' = \kappa_n^4 X_n$ to the following equations:

$$\sum_{n=1}^{\infty} [\mu_1 \ddot{y}_{1n} + K_{1n} y_{1n} - k_2 y_{2n}] X_n = 0, \tag{A-3}$$

$$\sum_{n=1}^{\infty} [\mu_2 \ddot{y}_{2n} + K_{2n} y_{2n} - k_2 y_{1n}] X_n = 0, \tag{A-4}$$

where

$$K_{1n} = k_1 + k_2 + \kappa_n^4 S_1, \qquad K_{2n} = k_2 + k_3 + \kappa_n^4 S_2. \tag{A-5}$$

Multiplying Eqs. (A-3) and (A-4) by the eigenfunction $X_j(x)$ and integrating with respect to x between 0 and the beam's length L leads under consideration of Eq. (11) to

$$\mu_1 \ddot{y}_{1n} + K_{1n} y_{1n} - k_2 y_{2n} = 0, \tag{A-6}$$

$$\mu_2 \ddot{y}_{2n} + K_{2n} y_{2n} - k_2 y_{1n} = 0. \tag{A-7}$$

Introducing Eq. (15) and the second time derivatives into Eqs. (A-6) and (A-7) leads to

$$\begin{bmatrix} K_{1n} - \mu_1 \omega_n^2 & -k_2 \\ -k_2 & K_{2n} - \mu_2 \omega_n^2 \end{bmatrix} \begin{bmatrix} Y_{1n} \\ Y_{2n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{A-8}$$

Setting the determinant of the coefficients matrix to zero yields:

$$\mu_1 \mu_2 \omega_n^4 - (K_{1n}\mu_2 + K_{2n}\mu_1)\omega_n^2 + K_{1n} K_{2n} - k_2^2 = 0. \tag{A-9}$$

Solving this equation yields the circular eigenfrequencies in the form:

$$\omega_{1,2n} = \left[\frac{1}{2\mu_1\mu_2} \left[K_{1n}\mu_2 + K_{2n}\mu_1 \mp \sqrt{(K_{1n}\mu_2 + K_{2n}\mu_1)^2 - 4\mu_1\mu_2(K_{1n}K_{2n} - k_2^2)} \right] \right]^{1/2}. \tag{A-10}$$