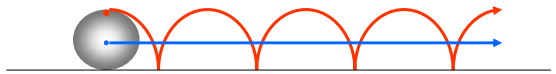
Chapter 11: Rolling, Torque and Angular Momentum

- Rolling motion
- Vector Products
- Torque
- Angular Momentum
- Conservation of Angular Momentum

Rolling Motion

- Rolling motion: A motion that is a combination of rotational and translational motion, e.g. a wheel rolling down the road.



- the center of the object moves in a straight line parallel to the surface (blue line) , while a point on the rim rotate about the center of mass (red line).

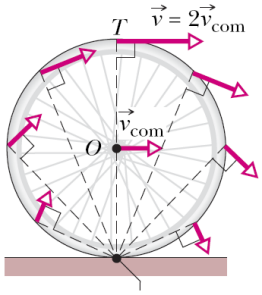
Rolling Motion

- Looking at the wheel as it rotate at angular speed ω about the point P in contacte with the surface → linear velocity v depend on the distance from rotational axis at P
- → While the angular velocity (and angular acceleration) of any point of the object is the same, the linear speed changes →

$$\vec{v}_P = 0$$

$$\vec{v}_O = r\omega = \vec{v}_{com}$$

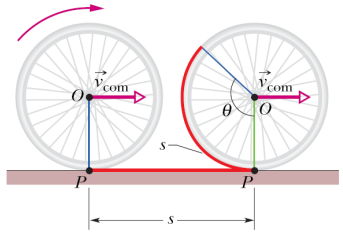
$$\vec{v}_T = 2r\omega = 2\vec{v}_{com}$$



Rolling Motion

- Pure rolling motion: object rotating at angular speed ω with No slipping occurs.
- as object rolls down, The center of mass will move a distance equal to arc length s where

$$s = R\theta$$

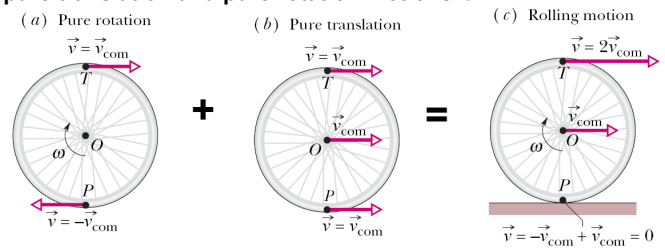


- The linear speed and linear acceleration of the center of mass can be found →

$$v_{com} = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega \quad \text{and} \quad a_{com} = \frac{dv_{com}}{dt} = R\frac{d\omega}{dt} = R\alpha$$

Rolling Motion

- Or Pure rolling motion can be considered as a superposition of **pure translation and pure rotation motions →**

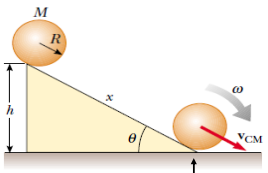


$$K = K_T + K_R$$

- Hence, Kinetic energy of rolling motion $K = \frac{1}{2}Mv_{com}^2 + \frac{1}{2}I_{com}\omega^2$

Rolling Motion: Example

A uniform ball, of mass M 6 kg and radius R, rolls from rest of height h = 1.2 m down a ramp at angle θ= 30°. (a) what is its speed at the bottom?



$$\Delta E = 0 \Rightarrow E_f = E_i$$

$$K_f + U_f = K_i + U_i$$

$$\left(\tfrac{1}{2}I_{com}\omega^2 + \tfrac{1}{2}Mv_{com}^2\right) + 0 = 0 + Mgh$$

with $I_{com} = \frac{2}{5}MR^2$ and $\omega = v_{com}/R$

$$\rightarrow \tfrac{1}{2}\left(\tfrac{2}{5}MR^2\right)(v_{com}/R)^2 + \tfrac{1}{2}Mv_{com}^2 = Mgh$$

$$\Rightarrow v_{com} = \sqrt{\left(\tfrac{10}{7}\right)gh} = \sqrt{\left(\tfrac{10}{7}\right)(9.8)(1.2)} = 4.10 \text{ m/s}$$

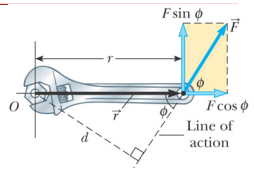
$K = K_R + K_T$

$$K = \left(\tfrac{1}{2}I_{com}\omega^2 + \tfrac{1}{2}Mv_{com}^2\right)$$

Torque (revisited)

- In chapter 10, the torque magnitude was defined by

$$\tau = rF\sin\phi$$

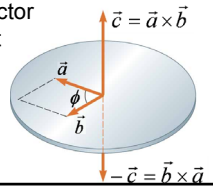


- From cross product definition → we can found the vector relation between $\vec{r}$ and $\vec{F}$

- In Chapter 3, the cross product states that For two vectors $\vec{a}$ and $\vec{b}$, the vector product is a third vector $\vec{c}$ so that

$$\vec{c} = \vec{a} \times \vec{b}$$

With magnitude $c = ab\sin\phi$

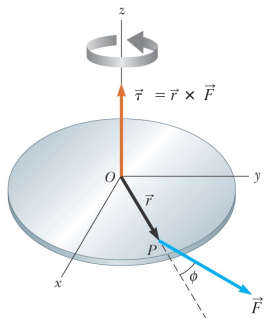



Right-hand rule

Torque

- Hence the torque can be defined as:

$$\vec{\tau} = \vec{r} \times \vec{F} \quad \text{with} \quad \tau = rF\sin\phi$$



Ex :

A force $\vec{F} = (2\hat{i} + 3\hat{j})N$ act on particle at position $\vec{r} = (7\hat{i} + 1\hat{j})m$.

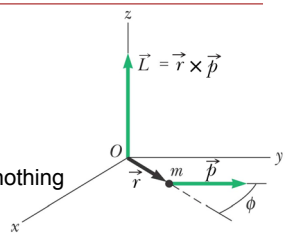
Find the toque on particle about the origin

$$\vec{\tau} = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 7 & 1 & 0 \\ 2 & 3 & 0 \end{vmatrix} = (0)\hat{i} - (0)\hat{j} + (21-2)\hat{k} = 19\hat{k}$$

Angular Momentum $\vec{L}$

The net torque $\sum \vec{\tau} = \vec{r} \times \sum \vec{F}$

but $\vec{F} = \dfrac{d\vec{p}}{dt} \Rightarrow \sum \vec{\tau} = \vec{r} \times \dfrac{d\vec{p}}{dt}$



If you add zero to the torque you change nothing

you can add $\vec{p} \times \dfrac{d\vec{r}}{dt} = 0$

because $\dfrac{d\vec{r}}{dt} = \vec{v}$ *that is // to* $\vec{p}$ $\Rightarrow \sum \tau = r \times \dfrac{dp}{dt} + p \times \dfrac{dr}{dt} = \dfrac{d(r \times p)}{dt}$

Where $\vec{L} = \vec{r} \times \vec{p} \quad (kgm^2/s)$

Is the angular momentum.

Hence, $\sum \vec{\tau} = \dfrac{d\vec{L}}{dt}$

Angular Momentum $\vec{L}$

$\vec{L} = \vec{r} \times \vec{p} \quad (kgm^2/s) \Rightarrow \vec{L}$ is a vector $\perp$ to both $\vec{r}$ and $\vec{p}$

With magnitude $L = rp\sin\theta = mvr\sin\theta$

For a system of particles, $\vec{L}_{tot} = \sum_i \vec{L}_i = \vec{L}_1 + \vec{L}_2 + \vec{L}_3 + ...\vec{L}_i$

Ex : A 2 kg particle moving at velocity $\vec{v} = (-3\hat{i} + 2\hat{k})m/s$ located at x - axis at position $x = 5m$. find the angumar momentum of the particle about the origin

$\vec{L} = \vec{r} \times \vec{p}$ with $\vec{p} = m\vec{v} = (-6\hat{i} + 4\hat{k})kg.m/s$ and $\vec{r} = 5\hat{i}\ m$

$$\Rightarrow \vec{L} = \vec{r} \times \vec{p} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 0 & 0 \\ -6 & 0 & 4 \end{vmatrix} = \hat{i}(0-0) - \hat{j}(20-0) + \hat{k}(0-0)$$

$\Rightarrow \vec{L} = -20\hat{j}\ kg.m^2/s$

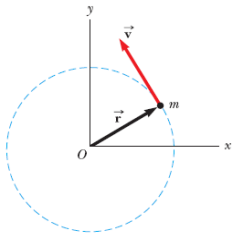
Angular Momentum $\vec{L}$: Example

A particle moves in the *xy* plane in a circular path of radius *r* as shown

Find the magnitude and direction of its angular momentum

$\vec{L} = \vec{r} \times \vec{p} = m\vec{r} \times \vec{v}$

$\vec{L} = (mvr)\hat{k}$ (θ=90°➔ sinθ = 1)



$L = mvr$. *But* $v = r\omega \Rightarrow$

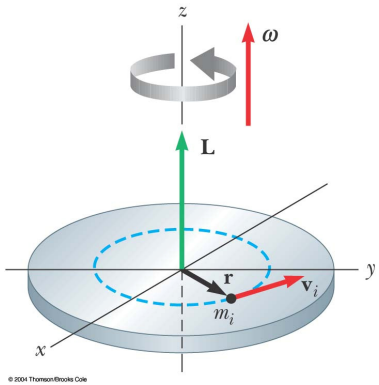
$L = mr^2\omega = I\omega$

Angular Momentum of a Rotating Rigid Object

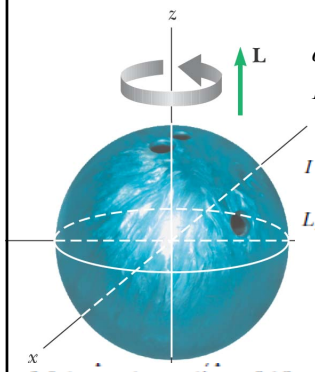
For an object rotating about the z-axis:

$L_z = I\omega$

The angular momentum and angular velocity are along the z-axis



Example: Bowling ball



$\omega = 10 rev/s$

$L = ?$

$L_z = I\omega$

$I = \frac{2}{5}MR^2 = \frac{2}{5}(6.0\text{ kg})(0.12\text{ m})^2 = 0.035\text{ kg}\cdot\text{m}^2$

$L_z = I\omega = (0.035\text{ kg}\cdot\text{m}^2)(10\text{ rev/s})(2\pi\text{ rad/rev})$
$= 2.2\text{ kg}\cdot\text{m}^2/\text{s}$

6.0 kg and a radius of 12 cm

Conservation of Angular Momentum

- The total angular momentum of a system is conserved if the net external torque acting on the system is zero.

$$L = \text{constant}$$

- The initial and final angular momentum of an isolated system is constant after an internal rearrangement.

$$L_i = L_f = cons\tan t$$

- For a rigid rotating object:

$$I_i\omega_i = I_f\omega_f = \text{constant}$$

Example

A horizontal platform in the shape of a circular disk rotates freely in a horizontal plane about a frictionless vertical axle (Fig. 11.11). The platform has a mass $M = 100$ kg and a radius $R = 2.0$ m. A student whose mass is $m = 60$ kg walks slowly from the rim of the disk toward its center. If the angular speed of the system is 2.0 rad/s when the student is at the rim, what is the angular speed when he reaches a point $r = 0.50$ m from the center?

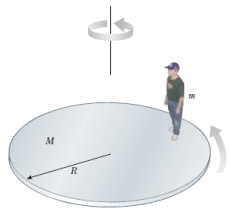
Example

$\mathbf{L}_{tot} = \text{constant}$ or $\mathbf{L}_i = \mathbf{L}_f$

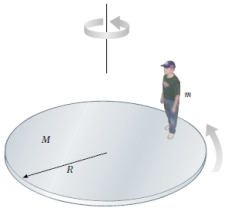
$$I_i\omega_i = I_f\omega_f$$

$$I_i = I_{pi} + I_{si} = \tfrac{1}{2}MR^2 + mR^2$$

$$I_f = I_{pf} + I_{sf} = \tfrac{1}{2}MR^2 + mr^2$$



Example



$$I_i\omega_i = I_f\omega_f$$

$$\left(\tfrac{1}{2}MR^2 + mR^2\right)\omega_i = \left(\tfrac{1}{2}MR^2 + mr^2\right)\omega_f$$

$$\omega_f = \left(\frac{\frac{1}{2}MR^2 + mR^2}{\frac{1}{2}MR^2 + mr^2}\right)\omega_i$$

$$= \left(\frac{\frac{1}{2}(100\text{ kg})(2.0\text{ m})^2 + (60\text{ kg})(2.0\text{ m})^2}{\frac{1}{2}(100\text{ kg})(2.0\text{ m})^2 + (60\text{ kg})(0.50\text{ m})^2}\right)(2.0\text{ rad/s})$$

$$= \left(\frac{440\text{ kg}\cdot\text{m}^2}{215\text{ kg}\cdot\text{m}^2}\right)(2.0\text{ rad/s}) = \boxed{4.1\text{ rad/s}}$$

As expected, the angular speed increases.