Chapter 11: Rolling, Torque and Angular Momentum

- Rolling motion
- Vector Products
- Torque
- Angular Momentum
- Conservation of Angular Momentum

Rolling Motion

- Rolling motion: A motion that is a combination of rotational and translational motion, e.g. a wheel rolling down the road.

- While the angular velocity (and angular acceleration) of any point of the object is the same, the linear speed changes

\[
\vec{v}_p = 0
\]

\[
\vec{v}_O = r\omega = \vec{v}_{com}
\]

\[
\vec{v}_r = 2r\omega = 2\vec{v}_{com}
\]

Looking at the wheel as it rotate at angular speed \(\omega\) about the point P in contacte with the surface \(\rightarrow\) linear velocity \(v\) depend on the distance from rotational axis at P

\(\rightarrow\) While the angular velocity (and angular acceleration) of any point of the object is the same, the linear speed changes \(\rightarrow\)

\[
\vec{v}_p = 0
\]

\[
\vec{v}_O = r\omega = \vec{v}_{com}
\]

\[
\vec{v}_r = 2r\omega = 2\vec{v}_{com}
\]

Rolling Motion

- Pure rolling motion: object rotating at angular speed \(\omega\) with No slipping occurs.

- as object rolls down, The center of mass will move a distance equal to arc length s where

\[
s = R\theta
\]

- The linear speed and linear acceleration of the center of mass can be found \(\rightarrow\)

\[
v_{com} = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega
\]

and

\[
a_{com} = \frac{dv_{com}}{dt} = R\frac{d\omega}{dt} = R\alpha
\]
Rolling Motion

- Pure rolling motion can be considered as a superposition of pure translation and pure rotation motions.

\[ \omega = \omega_{\text{com}} + \omega_{\text{rot}} \]

- Hence, Kinetic energy of rolling motion

\[ K = \frac{1}{2} M v_{\text{com}}^2 + \frac{1}{2} I_{\text{com}} \omega^2 \]

Torque (revisited)

- In Chapter 10, the torque magnitude was defined by

\[ \tau = rF \sin \phi \]

- From cross product definition → we can found the vector relation between \( \vec{r} \) and \( \vec{F} \)

\[ \vec{c} = \vec{a} \times \vec{b} \]

- In Chapter 3, the cross product states that

  For two vectors \( \vec{a} \) and \( \vec{b} \), the vector product is a third vector \( \vec{c} \) so that
  
  \( \vec{c} = \vec{a} \times \vec{b} \)

  With magnitude \( c = ab \sin \phi \)

Rolling Motion: Example

A uniform ball, of mass \( M = 6 \) kg and radius \( R \), rolls from rest of height \( h = 1.2 \) m down a ramp at angle \( \theta = 30^\circ \). (a) what is its speed at the bottom?

\[ \Delta E = 0 \Rightarrow E_f = E_i \]

\[ K_f + U_f = K_i + U_i \]

\[ \frac{1}{2} M R^2 \omega^2 + \frac{1}{2} M v_{\text{com}}^2 = 0 \]

\[ \omega = v_{\text{com}}/R \]

\[ K = \frac{1}{2} M R^2 + \frac{1}{2} I_{\text{com}} \omega^2 \]

\[ K = \frac{1}{2} M R^2 \]

\[ \frac{1}{2} M R^2 (v_{\text{com}}/R)^2 + \frac{1}{2} M v_{\text{com}}^2 = Mgh \]

\[ v_{\text{com}} = \sqrt{(\frac{10}{2})gh} = \sqrt{(\frac{10}{2})(9.8)(1.2)} = 4.10 \text{ m/s} \]

Torque

- Hence the torque can be defined as:

\[ \vec{\tau} = \vec{r} \times \vec{F} \quad \text{with} \quad \tau = rF \sin \phi \]

Ex:

A force \( \vec{F} = (2\hat{i} + 3\hat{j})N \) act on particle at position \( \vec{r} = (7\hat{i} + 1\hat{j})m \).

Find the toque on particle about the origin

\[ \vec{\tau} = \vec{r} \times \vec{F} = \begin{vmatrix} i & j & k \\ 7 & 1 & 0 \\ 2 & 3 & 0 \end{vmatrix} = (0\hat{i} - (0)\hat{j} + (21 - 2)\hat{k}) = 19\hat{k} \]
**Angular Momentum $\vec{L}$**

The net torque $\sum \vec{\tau} = \vec{r} \times \sum \vec{F}$

but $\vec{F} = \frac{d\vec{p}}{dt} \Rightarrow \sum \vec{\tau} = \vec{r} \times \frac{d\vec{p}}{dt}$

If you add zero to the torque you change nothing.

you can add $\vec{p} \times \frac{d\vec{r}}{dt} = 0$

because $\frac{dr}{dt} = \vec{v}$ that is // to $\vec{p}$

$\Rightarrow \sum \tau = \vec{r} \times \frac{d\vec{p}}{dt} + \vec{p} \times \frac{dr}{dt} = \frac{d(\vec{r} \times \vec{p})}{dt}$

Where $\vec{L} = \vec{r} \times \vec{p}$ (kgm²/s)

Hence, $\sum \vec{\tau} = \frac{d\vec{L}}{dt}$

**Angular Momentum $\vec{L}$: Example**

A particle moves in the $xy$ plane in a circular path of radius $r$ as shown.

Find the magnitude and direction of its angular momentum

$\vec{L} = \vec{r} \times \vec{p} = m\vec{r} \times \vec{v}$

$\vec{L} = (mr\vec{v})\hat{k}$ ($\theta = 90^\circ \Rightarrow \sin \theta = 1$)

$L = mrv$. But $v = r\omega$ $\Rightarrow$

$L = mrr^2\omega = I\omega$

**Angular Momentum of a Rotating Rigid Object**

For an object rotating about the z-axis:

$L_z = I\omega$

The angular momentum and angular velocity are along the z-axis.

Ex: A 2 kg particle moving at velocity $\vec{v} = (-3\hat{i} + 2\hat{k})$m/s located at $x$-axis at position $x = 5$ m. Find the angular momentum of the particle about the origin.

$\vec{L} = \vec{r} \times \vec{p}$ with $\vec{p} = m\vec{v} = (-6\hat{i} + 4\hat{k})$kg m/s and $\vec{r} = 5\hat{i}$ m

$\Rightarrow \vec{L} = \vec{r} \times \vec{p} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 0 & 0 \\ -6 & 0 & 4 \end{vmatrix} = \hat{i}(0-0) - \hat{j}(20-0) + \hat{k}(0-0)$

$\Rightarrow \vec{L} = -20\hat{j}$ kgm²/s
Example: Bowling ball

![Diagram of a bowling ball with angular momentum calculations]

Conservation of Angular Momentum

- The total angular momentum of a system is conserved if the net external torque acting on the system is zero.
  \[ L = \text{constant} \]

- The initial and final angular momentum of an isolated system is constant after an internal rearrangement.
  \[ L_i = L_f = \text{cons} \tan t \]

- For a rigid rotating object:
  \[ I_i \omega_i = I_f \omega_f = \text{constant} \]

Example

A horizontal platform in the shape of a circular disk rotates freely in a horizontal plane about a frictionless vertical axle (Fig. 11.11). The platform has a mass \( M = 100 \) kg and a radius \( R = 2.0 \) m. A student whose mass is \( m = 60 \) kg walks slowly from the rim of the disk toward its center. If the angular speed of the system is \( 2.0 \) rad/s when the student is at the rim, what is the angular speed when he reaches a point \( r = 0.50 \) m from the center?
Example

\[ I \omega_i = I \omega_f \]

\[ (\frac{1}{2}MR^2 + mR^2) \omega_i = (\frac{1}{2}MR^2 + mR^2) \omega_f \]

\[ \omega_f = \left( \frac{\frac{1}{2}MR^2 + mR^2}{\frac{1}{2}MR^2 + mR^2} \right) \omega_i \]

\[ = \left( \frac{\frac{1}{2}(100 \text{ kg})(2.0 \text{ m})^2 + (60 \text{ kg})(2.0 \text{ m})^2}{\frac{1}{2}(100 \text{ kg})(2.0 \text{ m})^2 + (60 \text{ kg})(0.50 \text{ m})^2} \right) (2.0 \text{ rad/s}) \]

\[ = \left( \frac{440 \text{ kg} \cdot \text{m}^2}{215 \text{ kg} \cdot \text{m}^2} \right) (2.0 \text{ rad/s}) = \frac{4.1 \text{ rad/s}}{1} \]

As expected, the angular speed increases.