Chapter 10: Rotation

- Angular position, velocity, acceleration
- Constant angular acceleration
- Angular and linear quantities
- Rotational kinetic energy
- Rotational inertia
- Torque
- Newton’s 2nd law for rotation
- Work and rotational kinetic energy

10.1: What is physics?

- In previous chapters we have discussed the translational motion
- In this chapter we will discuss the motion when object turn about an axis (Rotational Motion)
- Variables in rotational motion are analogous to those for translational motion with few changes
- We will:
  - Discuss the quantities in angular variables (we will focus on the angle when object rotating)
  - Find the angular position, velocity, and acceleration.
  - Apply Newton’s second law but instead of force and mass we will use torque and rotational inertia.
  - Apply energy concepts to angular quantities like work and rotational kinetic energy

10.2: Rotational variables

- We will focus on rotation of a rigid body about a fixed axis
  - Rigid body: body that can rotate with all its parts packed together without any change in its shape
  - Fixed axis: rotation about an axis that does not move
- Figure shows a rigid body of arbitrary shape in pure rotation about the z-axis of a coordinate system
  - Every point of the body moves in a circle whose center lies on the axis of rotation (see the arbitrary reference line), and every point moves through the same angle during a particular time interval

\[ \theta = \frac{s}{r} \]

Angular position in radians (rad.) or revolution

10.2: Rotational variables:

- Consider a particle on rigid object at point p rotates through an angle \( \theta \)
  - As object rotates, the point \( P \) make an Arc length \( s \rightarrow \theta = \frac{s}{r} \)
  - Radians: ratio between the two length \( s \) and \( r \) dimensionless quantity
- When \( r = s \rightarrow \theta \equiv 1 \text{ rad.} \equiv 57.3^\circ \)
- Full circle = 360° = 2\( \pi \) rad. = 1 revolution (2\( \pi \) rad).
- Where \( \pi = 22/7 = 3.14 \rightarrow \) we can convert between rad. and degree from the correlation:

\[
\begin{align*}
\text{Degree} & \quad \text{rad.} \\
360^\circ & \rightarrow 2\pi \\
\theta & \rightarrow \frac{2\pi}{360} \theta (\text{deg})
\end{align*}
\]
10.2: Rotational variables

Angular displacement: \[ \Delta \theta = \theta_f - \theta_i \] (Rad.)

Average angular speed: \[ \omega_{av} = \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta \theta}{t_f - t_i} \] (Rad/s)\( \equiv \) s\(^{-1}\)

Instantaneous angular speed: \[ \omega = \lim_{t \to \theta} \frac{\Delta \theta}{t_f - t_i} = \frac{d \theta}{dt} \] (Rad/s)\( \equiv \) s\(^{-1}\)

Average angular acceleration: \[ \alpha_{av} = \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta \omega}{t_f - t_i} \] (Rad/s\(^2\))\( \equiv \) s\(^{-2}\)

Instantaneous angular acceleration: \[ \alpha = \lim_{t \to \theta} \frac{\Delta \omega}{t_f - t_i} = \frac{d \omega}{dt} \] (Rad/s\(^2\))\( \equiv \) s\(^{-2}\)

All particles of a rigid object rotate at the same angular displacement, speed and acceleration.

10.2: Rotational variables: Example

Example: Rotation of a compact disk
A CD rotates at high speed. The disk has radius 6.0cm and while data is being read by the laser it spins at 7200rev/min. What is the CD’s angular velocity in radians per second. How much time is required for it to rotate through 90°? If it starts from rest and reaches full speed in 4.0s, what is its average angular acceleration?

\[ \omega = \frac{7200}{60} \text{ rev/min} = 720 \text{ rad/s} \]

\[ \Delta \theta = 90^\circ = \frac{90}{180} \pi \text{ rad} \]

\[ \alpha_{av} = \frac{\Delta \omega}{\Delta t} = \frac{\frac{720}{4.0} \text{ rad/s}}{4.0 \text{s}} = 45 \text{ rad/s}^2 \]

10.3: are angular quantities vectors?

- Angular velocity \( \omega \) is a vector \( \rightarrow \) it can be written as \( \vec{\omega} \)
- For rotation about a fixed axis, the direction of the angular velocity is along the axis of rotation.
- Use the right hand rule to determine direction.
- Also angular acceleration \( \alpha \) is a vector quantity \( \rightarrow \) can be written \( \vec{\alpha} \) having same rules for direction and same rules of speeding up rotation or slowing down rotation.
### 10.4: Rotation with constant angular acceleration

For rotational motion with constant rotational acceleration $\alpha$

The equations of motion are similar to the equation of motion in one dimension (1D);

Only do the following symbol replacement:

$x \equiv \theta$

$v \equiv \omega$

$a \equiv \alpha$

---

#### 10.4: Rotation with constant angular acceleration

**Linear (1D) Motion**

with constant linear acceleration, $a$

\[
\begin{align*}
    v &= v_0 + at \\
    x &= x_0 + \frac{1}{2} (v_0 + v)t \\
    \Delta x &= v_0 t + \frac{1}{2} at^2 \\
    v^2 &= v_0^2 + 2a\Delta x
\end{align*}
\]

**Rotational Motion**

with constant rotational acceleration, $\alpha$

\[
\begin{align*}
    \omega &= \omega_0 + \alpha t \\
    \theta &= \theta_0 + \frac{1}{2} (\omega_0 + \omega)t \\
    \Delta \theta &= \omega_0 t + \frac{1}{2} \alpha t^2 \\
    \omega^2 &= \omega_0^2 + 2\alpha \Delta \theta
\end{align*}
\]

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### 10.4: Rotation with constant angular acceleration: Example

A wheel rotates with a constant angular acceleration of 3.50 rad/s², $\alpha$

(A) If the angular speed of the wheel is 2.00 rad/s at $t = 0$, through what angular displacement does the wheel rotate in 2.00 s?

(B) Through how many revolutions has the wheel turned during this time interval?

(C) What is the angular speed of the wheel at $t = 2.00$ s?

\[
\begin{align*}
    \alpha &= 3.50 \text{ rad/s}^2, t_i = 0, \omega_i = 2.00 \text{ rad/s}, t_f = 2.00 \text{ s} \\
    \Delta \theta &= ?, \text{revolutions} = ?, \omega_f = ?
\end{align*}
\]

\[
\begin{align*}
    \Delta \theta &= \theta_f - \theta_i = \omega_f t + \frac{1}{2} \alpha t^2 \\
    &= (2.00 \text{ rad/s})(2.00 \text{ s}) + \frac{1}{2} (3.50 \text{ rad/s}^2)(2.00 \text{ s})^2 = 11.0 \text{ rad}
\end{align*}
\]

\[
\begin{align*}
    \Delta \theta &= 11 \text{ rad} \left( \frac{1 \text{ rev.}}{2\pi \text{ rad.}} \right) = 1.75 \text{ rev.}
\end{align*}
\]

\[
\begin{align*}
    \omega_f &= \omega_i + \alpha t = 2.00 \text{ rad/s} + (3.50 \text{ rad/s}^2)(2.00 \text{ s}) \\
    &= 9.00 \text{ rad/s}
\end{align*}
\]
10.4: Rotation with constant angular acceleration: Example

While you are operating a Rotor cylinder, the angular velocity of the cylinder from 3.4 rad/s to 2.0 rad/s in 20 rev., at constant angular acceleration. Find (a) the angular acceleration (b) time to decrease the angular speed.

We have

\[ \omega_f = 3.4 \text{ rad/s} \quad \omega_i = 2 \text{ rad/s} \quad \Delta \theta = 20 \text{ rev.} \]

\[ \omega_f^2 = \omega_i^2 + 2a\Delta \theta \quad \text{with} \quad \Delta \theta = 20 \text{ rev.}
\]

\[ 125.7 \text{ rad.} = 2\pi \text{ rad.} \cdot \frac{20 \text{ rev.}}{1 \text{ rev.}} \]

\[ \Rightarrow \alpha = \frac{\omega_f^2 - \omega_i^2}{2\Delta \theta} \]

\[ \Rightarrow \alpha = \frac{4 - 11.56}{251.4} = -0.03 \text{ rad/s}^2 \]

\[ \Rightarrow \alpha = \frac{4 - 11.56}{251.4} = -0.03 \text{ rad/s}^2 \]

\[ \Rightarrow \omega = \omega_i + \alpha t \]

\[ \Rightarrow t = \frac{2 - 3.4}{-0.03} = 46.7 \text{ s} \]

10.5: Relating The Linear And Angular Variables

For a rotating object, both linear and angular quantities are simultaneously exist → there must be a relation between them

- Arc length s:
  \[ s = r\theta \]

- Tangential speed of a point P:
  \[ v = r\omega \]

- Tangential acceleration of a point P:
  \[ a_t = r\alpha \]

- Centripetal acceleration for rotation object:
  \[ a_c = \frac{v^2}{r} = r\omega^2 \]

\[ \Rightarrow \text{Magnitude of total acceleration} \]

\[ a = \sqrt{a_t^2 + a_c^2} \]

A race car accelerates constantly from a speed of 40 m/s to 60 m/s in 5 s around a circular track of radius 400 m. When the car reaches a speed of 50 m/s find the (a) Angular speed (b) Centripetal acceleration, Tangential acceleration, and angular acceleration (d) The magnitude of the total acceleration.

We have

\[ v = 40 \text{ m/s} \Rightarrow v = \frac{40}{400} = 0.1 \text{ rad/s} \]

\[ v = 60 \text{ m/s} \Rightarrow v = \frac{60}{400} = 0.15 \text{ rad/s} \]

\[ \alpha = \frac{v}{r} = \frac{0.125 \text{ rad/s}}{400} \]

\[ \Rightarrow a = \frac{(50)^2}{400} = 6.25 \text{ m/s}^2 \quad \text{or} \quad a_c = r\omega^2 = 400(0.125)^2 = 6.25 \text{ m/s}^2 \]

10.5: Relating The Linear And Angular Variables: Example: continued from previous slide

We have

\[ v = 40 \text{ m/s} \Rightarrow \omega_1 = 0.1 \text{ rad/s} \quad v = 60 \text{ m/s} \Rightarrow \omega_2 = 0.15 \text{ rad/s} \]

\[ \alpha = \frac{v}{r} \Rightarrow \alpha = \frac{0.125 \text{ rad/s}}{400} = 0.01 \text{ rad/s}^2 \]

- Or from angular quantities we can find angular acceleration \( a \)

\[ \omega_f = \omega_i + \alpha \Rightarrow \alpha = \frac{\omega_f - \omega_i}{t} = \frac{0.15 - 0.1}{5} = 0.01 \text{ rad/s}^2 \]

\[ \Rightarrow a = r\alpha = 400(0.01) = 4 \text{ m/s}^2 \]

\[ \text{c) The total acceleration} \quad a = \sqrt{a_t^2 + a_c^2} = \sqrt{6.25^2 + 4^2} = 7.42 \text{ m/s}^2 \]
10.6: Kinetic Energy Of Rotation

A collection of \( n \) particles rotating about a fixed axis has a rotational kinetic energy of:

\[
K = \frac{1}{2} \sum_{i=1}^{n} m_i v_i^2 = \frac{1}{2} \sum_{i=1}^{n} m_i r_i^2 \omega^2
\]

where \( \omega \) is the angular speed, \( r_i \) is the distance from rotational axis and \( v_i \) is the linear speed for \( i \) particle.

Mathematically, similar in shape to linear K with the following replacements:

\[
I \equiv m, \quad \omega \equiv v
\]

where \( I \) is the moment of inertia or rotational inertia:

\[
I = \sum_{i} m_i r_i^2 \quad \text{(kg.m}^2\text{)} \quad \text{(for collection of particles)}
\]

10.6: Kinetic Energy Of Rotation: Example

Four tiny spheres are fastened to the ends of two rods of negligible mass lying in the xy plane.

(A) If the system rotates about the y axis with an angular speed \( \omega \) find the moment of inertia and rotational kinetic energy about this axis:

\[
I_y = \sum m_i r_i^2 = M a^2 + M a^2 = 2 M a^2
\]

Spheres of mass \( m \) has \( I=0 \) because \( r=0 \), they lie on y-axis:

\[
K_y = \frac{1}{2} I \omega^2 = \frac{1}{2}(2 M a^2) \omega^2 = M a^2 \omega^2
\]

10.6: Kinetic Energy Of Rotation: Example: continued from previous slide

Four tiny spheres are fastened to the ends of two rods of negligible mass lying in the xy plane.

(B) Suppose the system rotates in the xy plane about an axis (the z axis) through \( O \) find the moment of inertia and kinetic energy about this axis:

\[
I_z = \sum m_i \Delta r_i^2 = M a^2 + M a^2 + 2 m b^2 + m b^2 = 2 M a^2 + 2 m b^2
\]

\[
K_R = \frac{1}{2} I \omega^2 = \frac{1}{2}(2 M a^2 + 2 m b^2) \omega^2 = (M a^2 + m b^2) \omega^2
\]

10.7: Calculating the rotational inertia

- For collection of particles, we had:

\[
I = \sum_{i} m_i r_i^2
\]

- For an extended, rigid object:

\[
I = \int \Delta m r_i^2 \quad \Rightarrow \quad I = \int r^2 dm
\]

- If \( dm = \rho dV \):

\[
m = \rho V \quad \Rightarrow \quad dm = \rho dV
\]

\[
m = \sigma A \quad \Rightarrow \quad dm = \sigma dA
\]

\[
m = \lambda L \quad \Rightarrow \quad dm = \lambda dL
\]

\[
I = \int \rho r^2 dV
\]
10.7: Calculating the rotational inertia: Example

Calculate the moment of inertia of a uniform rigid rod of length \( L \) and mass \( M \) about an axis perpendicular to the rod (the \( y \)-axis) and passing through its center of mass.

Extended object \( \Rightarrow I = \int r^2 \, dm \)

\[ dm = \lambda \, dx = \frac{M}{L} \, dx \; ; \; \lambda = \text{constant} \]

and \( r^2 = x^2 \)

\[ I = \int x^2 \, dx = \frac{M}{L} \int x^2 \, dx \]

\[ I = I_{\text{CM}} = \frac{M \, x^2}{L} \left[ -\frac{1}{2} \right] = \frac{1}{12} ML^2 \]

10.7: Calculating the rotational inertia: Parallel axis theorem

If \( I_{\text{CM}} \) is known, the moment of inertia through a parallel axis of rotation a distance \( h \) away from the center of mass is:

\[ I = I_{\text{CM}} + Mh^2 \]

- Long thin rod with rotation axis through center
  \[ I_{\text{CM}} = \frac{1}{12} ML^2 \]

- Long thin rod with rotation axis through end
  \[ I = \frac{1}{3} ML^2 \]

10.8: Torque

\[ \tau = rF \sin \phi \]

- Consider a rigid object about a pivot point (نقطة تركزة).
- A force is applied to the object.
- This force causes the object to rotate having what is called Torque \( \tau \).
10.8: Torque

Consider a particle of mass \( m \) rotating about a fixed axis under the influence of an applied force \( F \):
- The component \( F \) does no torque since (anti-parallel to \( r \)).
- The tangent component \( F \) has a torque \( \tau = Fr \) (sin 90° = 1).

But \( F = ma \), and \( a = r\alpha \).

\[ \Rightarrow \tau = Fr = m r^2 \alpha \]

\[ \Rightarrow \tau = I\alpha \]  

If more than one force applied to the object, \[ \sum \tau = \tau_{net} = I\alpha \]  

Newton's second law in rotation.

10.8: Torque: Example

A one-piece cylinder is shaped as shown. Two forces \( T_1 \) and \( T_2 \) are applied as shown.

(A) What is the net torque acting on the cylinder about the rotation axis?

\[ \sum \tau = 2 \tau_1 \tau_2 - 2 \tau_1 \]

(B) Suppose \( T_1 = 5.0 \text{ N}, \quad R_1 = 1.0 \text{ m}, \quad T_2 = 15.0 \text{ N}, \quad \text{and} \quad R_2 = 0.50 \text{ m}. \) What is the net torque about the rotation axis?

\[ \sum \tau = (15 \text{ N})(0.50 \text{ m}) - (5.0 \text{ N})(1.0 \text{ m}) = 2.5 \text{ N}\cdot\text{m} \]

Because this torque is positive, the cylinder will begin to rotate in the counterclockwise direction.

10.8: Example: A uniform rod of length \( L \) and mass \( M \) is attached as shown. The rod is released from rest in the horizontal position. What are the initial angular acceleration of the rod and the initial translational acceleration of its right end?

\[ \alpha = \frac{g}{L/2} \]

The translational acceleration is

\[ a_t = L\alpha = \frac{3g}{2L} \]

For rotation clockwise (الساعة) \( -\alpha \)  

For rotation counterclockwise (عكس عقارب الساعة) \( +\alpha \)  

The rode will move like a pendulum under the effect of \( F_g\).

• Extended object \( \Rightarrow \) look at the CM solution

\[ \tau = rF \sin \phi = rF \left( \frac{L}{2} \right)Mg \]

\[ \Rightarrow \tau = I\alpha \left( \frac{L}{2} \right)Mg \]

\[ \Rightarrow \alpha = \frac{\tau}{I} = \frac{(L/2)Mg}{1/3ML^2} = \frac{3g}{2L} \]  

The translational acceleration is

\[ a_t = L\alpha = \frac{3g}{2L} \]

- the tangent component \( F \) has a torque \( \tau = Fr \) (sin 90° = 1).

But \( F = ma \), and \( a = r\alpha \).

\[ \Rightarrow \tau = Fr = m r^2 \alpha \]

\[ \Rightarrow \tau = I\alpha \]  

If more than one force applied to the object, \[ \sum \tau = \tau_{net} = I\alpha \]  

Newton's second law in rotation.
10.8: Work and Rotational Kinetic Energy

Work in linear motion:
\[ dW = F \cdot ds \]
\[ W = \int F \cdot ds \]
\[ \theta = \frac{dW}{dt} = \vec{F} \cdot \vec{v} \]

Work in rotational motion:
\[ dW = F \cdot d\theta \]
\[ W = \tau \Delta \theta \]
\[ \theta = \tau \omega \]

10.8: Work and Rotational Kinetic Energy: Example

In previous example of disk, if the disk start from rest at time \( t = 0 \). What is its rotational kinetic energy \( K_r \) at \( t = 2.5 \) s?

From previous example we have:
\[ I = \frac{1}{2} MR^2 = \frac{1}{2} (2.5)(0.2)^2 = 0.05 \text{ kg} \cdot \text{m}^2 \]
\[ a = 4.8 \text{ m/s}^2, \text{ T} = 6 \text{ N}, \text{ and } a = 24 \text{ rad/s}^2 \]
\[ K_r = \frac{1}{2} I \omega^2 \quad \text{We need to find } \omega \text{ at } t = 2.5 \text{s} \]
\[ \omega = \omega_0 + at = 0 + 24(2.5) = 60 \text{ rad/s} \]
\[ \Rightarrow K_r = \frac{1}{2} I \omega^2 = \frac{1}{2} (0.05)(60)^2 = 90 \text{ J} \]

\[ M = 2.5 \text{ kg}, \text{ radius } R = 20 \text{ cm}, \text{ and } m = 1.2 \text{ kg} \]

10.8: Work and Rotational Kinetic Energy: Example: continued from previous slide

\[ \text{or } \Delta K_r = W \]
\[ K_f - K_i = K - 0 = \tau (\Delta \theta) \]

We need to find \( \tau \) and \( \Delta \theta \)

\[ \tau = RT = (0.2)(6) = 1.2 \text{ N} \cdot \text{m} \]
\[ \Delta \theta = \omega_0 t + \frac{1}{2} at^2 = 0 + \frac{1}{2} (24)(2.5)^2 = 75 \text{ rad} \]
\[ \Rightarrow K_r = \tau \Delta \theta = (1.2)(75) = 90 \text{ J} \]

\[ M = 2.5 \text{ kg}, \text{ radius } R = 20 \text{ cm}, \text{ and } m = 1.2 \text{ kg} \]
Review

- Linear quantities have analogous angular counterparts.

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Table 10-3

Some Corresponding Relations for Translational and Rotational Motion

- Object rotating make both linear and angular quantities at same instant → there is a relation with angular and linear quantities
- Torque is the tendency of a force to rotate an object.
- The total kinetic energy of a rotating object has to include its rotational kinetic energy.