5.8 Flexible Cables


- Examples: suspension bridges, transmission lines, messenger cables for supporting heavy trolley or telephone lines.

- To determine for design purposes: Tension force \( T \), span \( L \), sag \( h \), length of the cable \( s \).

- Assume: any resistance offered to bending is negligible. means: the tension force in the cable is always in the direction of the cable.

- Flexible cables may support
  - concentrated loads.
  - distributed loads
  - its own weight only
  - all three or only two of the above

- In several cases the weight of the cable may be negligible compared with the loads it supports.
General Relationships

- Assume:
  - the distributed load $w$ (in N/m) is *homogeneous* and has a *constant* thickness.
  - distributed load $w = w(x)$.

- The resultant $R$ of the vertical loading $w(x)$ is

$$ R = \int dR = \int wdx \quad (1) $$

- Position of $R$

$$ x_G = \frac{\int x dR}{R} \quad (2) $$

$x_G$ center of gravity, which equals the centroid of the area if $w$ is homogeneous.
Static Equilibrium

Note that the changes in both \( T \) and \( \theta \) are taken to be positive with a positive change in \( x \)

\[ \uparrow : \sum F_y = 0 \]
\[ (T + dT) \sin(\theta + d\theta) = T \sin \theta + \omega dx \]  \hspace{1cm} (3)

\[ \rightarrow : \sum F_x = 0 \]
\[ (T + dT) \cos(\theta + d\theta) = T \cos \theta \]  \hspace{1cm} (4)

With the equalities:

\[ \sin(a \pm b) = \sin a \cos b \pm \cos a \sin b \]
\[ \cos(a \pm b) = \cos a \cos b \mp \sin a \sin b \]

and the substitutions \( \sin(d\theta) = d\theta, \cos(d\theta) = 1 \), which hold in the limit as \( d\theta \)
approaches zero, yields

\[ (T + dT)(\sin \theta + \cos \theta d\theta) = T \sin \theta + \omega dx \]  \hspace{1cm} (5)
\[ (T + dT)(\cos \theta - \sin \theta d\theta) = T \cos \theta \]  \hspace{1cm} (6)

Neglecting the second-order term \( (dT d\theta) \) and simplifying leads to

\[ T \cos \theta \, d\theta + dT \sin \theta = \omega dx \]  \hspace{1cm} (7)
\[ -T \sin \theta \, d\theta + dT \cos \theta = 0 \]  \hspace{1cm} (8)
which can be written in the form

\[ d(T \sin \theta) = wx \]  \hspace{1cm} (9)
\[ d(T \cos \theta) = 0 \]  \hspace{1cm} (10)

Equation (10) means that the horizontal component of \( T \) remains constant.

\[ T \cos \theta = T_H = \text{const.} \]  \hspace{1cm} (11)

\[ \rightarrow T = \frac{T_H}{\cos \theta} \]  \hspace{1cm} (12)

Substituting Eq. (12) into Eq. (9) yields

\[ d(T_H \tan \theta) = wx \]  \hspace{1cm} (13)

with \( \tan \theta = \frac{dy}{dx} \), Eq. (13) becomes

\[ \frac{d^2y}{dx^2} = \frac{w}{T_H} \]  \hspace{1cm} (14)

which represents the differential equation for the flexible cable.

The solution of this equation with considering the boundary conditions yields the shape of the cable \( y = y(x) \).
**Parabolic Cable**

Assume: \( w = \text{const.}, \) load homogeneous

Example: suspension bridge

mass of the cable \( \ll \) mass of the bridge \( \rightarrow \) neglect the cable mass

Note: The mass of the cable itself is not distributed uniformly with the horizontal \((x\text{-axis})\).

- Place the coordinate origin at the lowest point of the cable.

\[
\frac{d^2 y}{dx^2} = \frac{w}{T_H} \tag{14}
\]

Integrating yields

\[
\frac{dy}{dx} = \frac{wx}{T_H} + C_1 \tag{15}
\]

\[
y(x) = \frac{wx^2}{2T_H} + C_1 x + C_2 \tag{16}
\]

Boundary conditions:

a) \( x = 0, \quad \frac{dy}{dx} = 0 \quad \text{Eq.} \ (15) \rightarrow C_1 = 0 \)

b) \( x = 0, \quad y = 0 \quad \text{Eq.} \ (16) \rightarrow C_2 = 0 \)

\[ \rightarrow \quad y(x) = \frac{wx^2}{2T_H} \tag{17} \]
Horizontal tension force $T_H$

At the lowest point, the tension force is horizontal.

BC: at $x = l_A, y = h_A$

Substituting this boundary condition into Eq. (17) gives

$$h_A = \frac{wl_A^2}{2T_H}$$

$$\rightarrow T_H = \frac{wl_A^2}{2h_A}$$

Note that $T_H$ is the minimum tension force in the cable ($T_H = T_{\text{min}}$).

Tension force $T(x)$

From the figure we get

$$T = \sqrt{T_H^2 + w^2x^2}$$

Where $T$ becomes maximum for $x = x_{\text{max}}$, since $T_H$ and $w$ are constants.

Using Eq. (19) yields

$$T = w\sqrt{x^2 + (l_A^2 + 2h_A)^2}$$

The maximum tension force occurs at $x = x_{\text{max}}$, in this case $x_{\text{max}} = l_A$.

$$T_{\text{max,}A} = wl_A\sqrt{1 + (l_A/2h_A)^2}$$
The length of cable \((s)\)

Length \(s_A\)

Integrating the differential length

\[
d s = \sqrt{(d x)^2 + (d y)^2}
\]  

(23)

gives

\[
\int_0^{s_A} d s = \int_0^{l_A} \sqrt{1 + (d y/d x)^2} d x
\]  

(24)

\[\text{a) exact solution}\]

\[
s_A = \frac{1}{2a} \left[ x \sqrt{x^2 + a^2} + a^2 \ln (x + \sqrt{x^2 + a^2}) \right]_{0}^{l_A}
\]

\[
s_A = \frac{1}{2a} \left[ l_A \sqrt{l_A^2 + a^2} + a^2 \ln \left( l_A + \sqrt{l_A^2 + a^2} \right) - a^2 \ln a \right]
\]  

(25)

where

\[
a = \frac{\tau H}{w} = \frac{l_A^2}{2 h_A}
\]  

(26)

\[\text{b) approximate solution}\]

using the binomial series

\[
(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \ldots
\]  

(27)

which converges for \(x^2 < 1\), and replacing \(x\) by \((w x / T_H)^2\) and setting \(n = 1/2\), we get

\[
s_A = l_A \left[ 1 + \frac{2}{3} \left( \frac{h_A}{l_A} \right)^2 - \frac{2}{5} \left( \frac{h_A}{l_A} \right)^4 + \ldots \right]
\]  

(28)
This series is convergent for values of \( h_A/l_A < 1/2 \), which holds for most practical cases.

For the cable section from the origin to \( B \) (\( x \) rotated \( 180^\circ \)), we obtain in a similar manner by replacing \( h_A, l_A \) and \( s_A \) by \( h_B, l_B \) and \( s_B \), respectively

\[
T_H = \frac{wl_B}{2h_B}
\]  \hspace{1cm} (29)

\[
T = w\sqrt{x^2 + (l_B^2 + 2h_B)^2}
\]  \hspace{1cm} (30)

\[
T_{max,B} = wl_B\sqrt{1 + (l_B/2h_B)^2}
\]  \hspace{1cm} (31)

\[
s_B = \frac{1}{2a}\left[l_B\sqrt{l_B^2 + a^2} + a^2 \ln\left(l_B + \sqrt{l_B^2 + a^2}\right) - a^2 \ln a\right]
\]  \hspace{1cm} (32)

where in this case

\[
a = \frac{l_B^2}{2h_B}
\]  \hspace{1cm} (33)

Approximate solution

\[
s_B = l_B\left[1 + \frac{2}{3} \left(\frac{h_B}{l_B}\right)^2 - \frac{2}{5} \left(\frac{h_B}{l_B}\right)^4 + \ldots\right]
\]  \hspace{1cm} (34)

Since \( h_A > h_B \), the absolute maximum tension force in the cable will naturally occur at end \( A \), since this side of the cable supports the greater proportion of the load.
**Symmetric case**

\[ s_A = s_B, \quad l_A = l_B, \quad h_A = h_B \]

total span \( L = 2l_A \), total sag \( h = h_A \)

In this case we get

\[ T_{\text{max}} = \frac{wL}{2} \sqrt{1 + \left(\frac{L}{4h}\right)^2} \]  \hfill (35)

\[ s = 2h \left[ \sqrt{1 + b^2} + b^2 \ln \left( \frac{L}{2} + \frac{L}{2} \sqrt{1 + b^2} \right) - b^2 \ln \left( \frac{L^2}{8h} \right) \right] \]  \hfill (36)

where

\[ b = \frac{L}{4h} \]  \hfill (37)

Approximate solution:

\[ s = L \left[ 1 + \frac{8}{3} \left( \frac{h}{L} \right)^2 - \frac{32}{5} \left( \frac{h}{L} \right)^4 + \cdots \right] \]  \hfill (38)

This series converges for all values of \( h/L < 1/4 \). In most cases \( h << L/4 \).

→ The first three terms of series (38) give a sufficiently accurate approximation.
Catenary Cable

Consider cable weight only

\[ wx \rightarrow \mu s; \quad wdx \rightarrow \mu ds \]

where \( \mu \) is the weight per unit length of the cable in N/m.

Eq. (10): \( T_H = T \cos \theta \quad \rightarrow \quad T = \frac{T_H}{\cos \theta} \)

Substituting into Eq. (9) and replacing \( wdx \) with \( \mu ds \) yields

\[ d(T_H \tan \theta) = \mu ds, \quad \tan \theta = \frac{dy}{dx} \]

\[ \rightarrow \quad d\left( T_H \frac{dy}{dx} \right) = \mu ds \quad (39) \]

Differentiation with respect to \( x \) yield

\[ \frac{d^2y}{dx^2} = \frac{\mu}{T_H} \frac{ds}{dx} \quad (40) \]

With \( ds = \sqrt{(dx)^2 + (dy)^2} \), we get

\[ \frac{d^2y}{dx^2} = \frac{\mu}{T_H} \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \quad (41) \]
Substitution: \( p = \frac{dy}{dx} \rightarrow \frac{dp}{dx} = \frac{d^2y}{dx^2} \) \hspace{1cm} (42)

\[ \rightarrow \frac{dp}{\sqrt{1+p^2}} = \frac{dx}{c} \] \hspace{1cm} (43)

where \( c = \frac{T_H}{\mu} \)

Substituting \( p = \sinh u, \ dp = \cosh u \ du, \ u = \arcsinh p \), gives

\[ \frac{dx}{c} = \frac{\cosh u}{\sqrt{1+\sinh^2 u}} \ du \] \hspace{1cm} (44)

Integrating leads to

\[ \frac{x}{c} = u + C_1 \] \hspace{1cm} (45)

Boundary condition:

At \( x = 0 \), \( \frac{dy}{dx} = p = 0 \rightarrow \sinh(0) = 0 \rightarrow C_1 = 0 \)

So that

\[ u = \frac{x}{c} = \arcsinh p \] \hspace{1cm} (46)

or

\[ p = \frac{dy}{dx} = \sinh \frac{x}{c} \] \hspace{1cm} (47)

which leads to

\[ dy = \sinh \frac{x}{c} \ dx \] \hspace{1cm} (48)

Integrating yields

\[ y = c\cosh \frac{x}{c} + C_2 \] \hspace{1cm} (49)
Boundary condition:

At \( x = 0, y = 0 \quad \rightarrow C_2 = -c \)

Thus, we obtain the equation of the curve formed by the cable

\[
y = \frac{T_H}{\mu} \left( \cosh \frac{\mu x}{T_H} - 1 \right)
\]  

(50)

**Cable length**

From the free-body diagram shown in the figure we see that

\[
\frac{dy}{dx} = \tan \theta = \frac{\mu s}{T_H} \quad \rightarrow s = \frac{T_H}{\mu} \frac{dy}{dx}
\]

Using Eq. (47), we get then

\[
s = \frac{T_H}{\mu} \sinh \frac{\mu x}{T_H}
\]  

(51)

Where the unknown minimum tension force \( T_H \) may be obtained from Eq. (50) by using the boundary condition \( y = h_A \) at \( x = l_A \).

**Tension force**

From the figure, we get

\[
T^2 = \mu^2 s^2 + T_H^2
\]  

(52)

Substituting Eq. (51) into Eq. (52) leads to
\[ T^2 = T_H^2 \left( 1 + \sinh^2 \frac{\mu x}{T_H} \right) = T_H^2 \cosh^2 \frac{\mu x}{T_H} \]  

(53)

or

\[ T = T_H \cosh \frac{\mu x}{T_H} \]  

(54)

With equation (50) we get

\[ T = T_H + \mu y \]  

(55)

The solution of catenary problems where the sag-to-span ratio is small may be approximated by the relations developed for the parabolic cable. A small sag-to-span ratio means a tight cable, and the uniform distribution of weight along the cable is not very different from the same load intensity distributed uniformly along the horizontal.