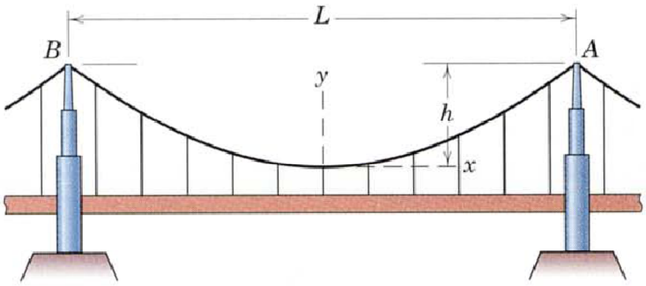
5.8 Flexible Cables

J. L. Meriam and L. G. Kraige, Engineering Mechanics, Statics 5th ed., SI

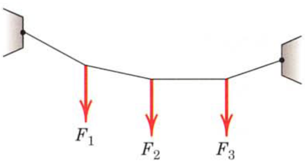
- Examples: suspension bridges, transmission lines, messenger cables for supporting heavy trolley or telephone lines.

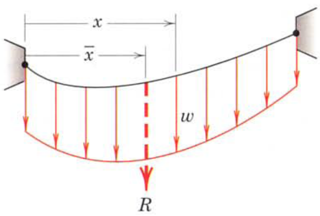
- To determine for design purposes : Tension force (T), span (L), sag (h), length of the cable (s).

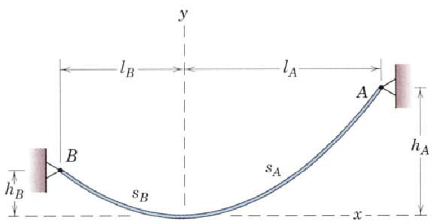


- Assume: any resistance offered to bending is negligible.
  means: the tension force in the cable is *always* in the direction of the cable.

- Flexible cables may support
  - concentrated loads.
  - distributed loads
  - its own weight only
  - all three or only two of the above



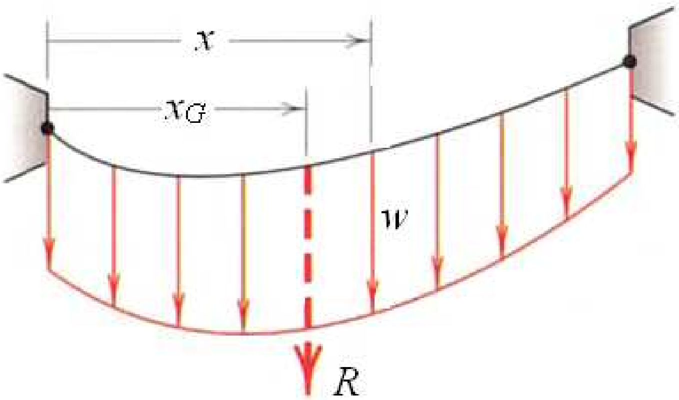




- In several cases the weight of the cable may be negligible compared with the loads it supports.

General Relationships

- Assume:
  - the distributed load w (in N/m) is *homogeneous* and has a *constant* thickness.
  - distributed load $w = w(x)$.



- The resultant **R** of the vertical loading $w(x)$ is

$$R = \int dR = \int w dx \tag{1}$$

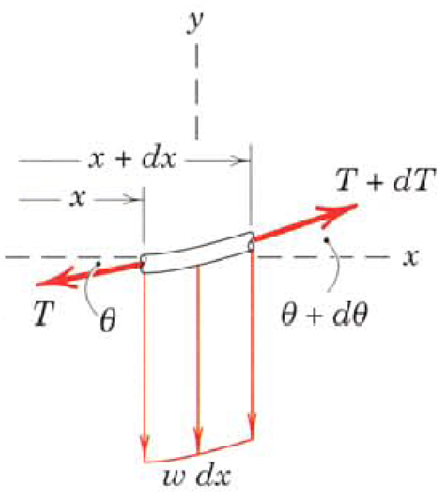
- Position of **R**

$$x_G = \frac{\int x dR}{R} \tag{2}$$

x_G center of gravity, which equals the centroid of the area if w is homogeneous.

Static Equilibrium

Note that the changes in both T and θ are taken to be positive with a positive change in x



$\uparrow\ :\sum F_y = 0$

$$(T + dT)\sin(\theta + d\theta) = T\sin\theta + wdx \tag{3}$$

$\rightarrow : \sum F_x = 0$

$$(T + dT)\cos(\theta + d\theta) = T\cos\theta \tag{4}$$

With the equalities:

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$$

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$

and the substitutions sin(dθ) = dθ, cos(dθ) =1, which hold in the limit as dθ approaches zero, yields

$$(T + dT)(\sin\theta + \cos\theta\, d\theta) = T\sin\theta + wdx \tag{5}$$

$$(T + dT)(\cos\theta - \sin\theta\, d\theta) = T\cos\theta \tag{6}$$

Neglecting the second-order term (dTdθ) and simplifying leads to

$$T\cos\theta\, d\theta + dT\sin\theta = wdx \tag{7}$$

$$-T\sin\theta\, d\theta + dT\cos\theta = 0 \tag{8}$$

which can be written in the form

$$d(T\sin\theta) = wdx \tag{9}$$

$$d(T\cos\theta) = 0 \tag{10}$$

Equation (10) means that the horizontal component of *T* remains constant.

$$T\cos\theta = T_H = const. \tag{11}$$

$$\rightarrow T = \frac{T_H}{\cos\theta} \tag{12}$$

Substituting Eq. (12) into Eq. (9) yields

$$d(T_H\tan\theta) = wdx \tag{13}$$

with $\tan\theta = \frac{dy}{dx}$, Eq. (13) becomes

$$\frac{d^2y}{dx^2} = \frac{w}{T_H} \tag{14}$$

which represents the differential equation for the flexible cable.

The solution of this equation with considering the boundary conditions yields the shape of the cable $y = y(x)$.

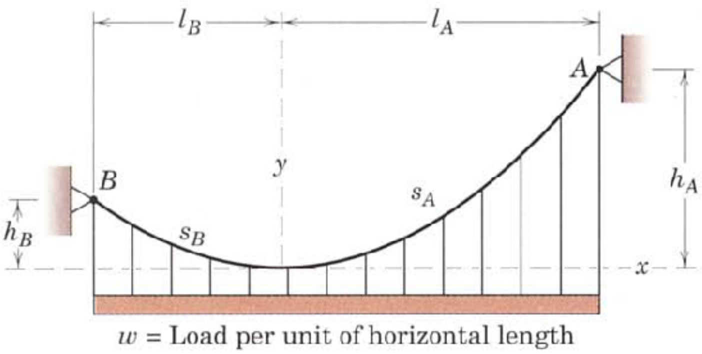
Parabolic Cable

Assume: w = const., load homogeneous

Example: suspension bridge

mass of the cable << mass of the bridge $\rightarrow$ neglect the cable mass

Note: The mass of the cable itself is not distributed uniformly with the horizontal (x-axis).



- Place the coordinate origin at the lowest point of the cable.

$$\frac{d^2y}{dx^2} = \frac{w}{T_H} \tag{14}$$

Integrating yields

$\frac{dy}{dx} = \frac{wx}{T_H} + C_1$ (15)

$$y(x) = \frac{wx^2}{2T_H} + C_1x + C_2 \tag{16}$$

Boundary conditions:

a) $x = 0$, $\frac{dy}{dx} = 0$ Eq. (15) $\rightarrow$ $C_1 = 0$
b) $x = 0$, $y = 0$ Eq. (16) $\rightarrow$ $C_2 = 0$

$$\rightarrow \quad y(x) = \frac{wx^2}{2T_H} \tag{17}$$

Horizontal tension force T_H

At the lowest point, the tension force is horizontal.

BC: at $x = l_A$, $y = h_A$

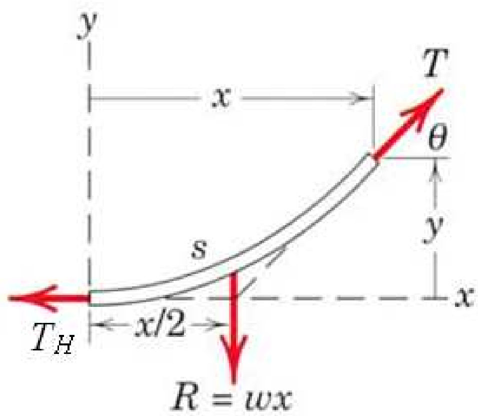
Substituting this boundary condition into Eq. (17) gives

$$h_A = \frac{w l_A^2}{2T_H} \tag{18}$$

$$\rightarrow \quad T_H = \frac{w l_A^2}{2h_A} \tag{19}$$

Note that T_H is the minimum tension force in the cable ($T_H = T_{min}$).

Tension force $T(x)$



From the figure we get

$$T = \sqrt{T_H^2 + w^2x^2} \tag{20}$$

Where T becomes maximum for $x = x_{max}$, since T_H and w are constants.

Using Eq. (19) yields

$$T = w\sqrt{x^2 + (l_A^2 + 2h_A)^2} \tag{21}$$

The maximum tension force occurs at $x = x_{\max}$; in this case $x_{max} = l_A$.

$$T_{max,A} = w l_A \sqrt{1 + (l_A/2h_A)^2} \tag{22}$$

The length of cable (s)

Length s_A

Integrating the differential length

$$ds = \sqrt{(dx)^2 + (dy)^2} \tag{23}$$

gives

$$\int_0^{s_A} ds = \int_0^{l_A} \sqrt{1 + (dy/dx)^2}\, dx \tag{24}$$

a) exact solution

$$s_A = \frac{1}{2a}\left[x\sqrt{x^2 + a^2} + a^2 \ln\left(x + \sqrt{x^2 + a^2}\right)\right]_0^{l_A}$$

$$s_A = \frac{1}{2a}\left[l_A\sqrt{l_A^2 + a^2} + a^2 \ln\left(l_A + \sqrt{l_A^2 + a^2}\right) - a^2 lna\right] \tag{25}$$

where

$$a = \frac{T_H}{w} = \frac{l_A^2}{2h_A} \tag{26}$$

b) approximate solution

using the binomial series

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots \tag{27}$$

which converges for $x^2 < 1$, and replacing x by $(wx/T_H)^2$ and setting $n = 1/2$, we get

$$s_A = l_A\left[1 + \frac{2}{3}\left(\frac{h_A}{l_A}\right)^2 - \frac{2}{5}\left(\frac{h_A}{l_A}\right)^4 + \cdots\right] \tag{28}$$

This series is convergent for values of $h_A/l_A<1/2$, which holds for most practical cases.

For the cable section from the origin to B (x rotated 180°), we obtain in a similar manner by replacing h_A, l_A and s_A by h_B, l_B and s_B, respectively

$$T_H = \frac{wl_B^2}{2h_B} \tag{29}$$

$$T = w\sqrt{x^2 + (l_B^2 + 2h_B)^2} \tag{30}$$

$$T_{max,B} = wl_B\sqrt{1 + (l_B/2h_B)^2} \tag{31}$$

$$s_B = \frac{1}{2a}\left[l_B\sqrt{l_B^2 + a^2} + a^2 \ln\left(l_B + \sqrt{l_B^2 + a^2}\right) - a^2 lna\right] \tag{32}$$

where in this case

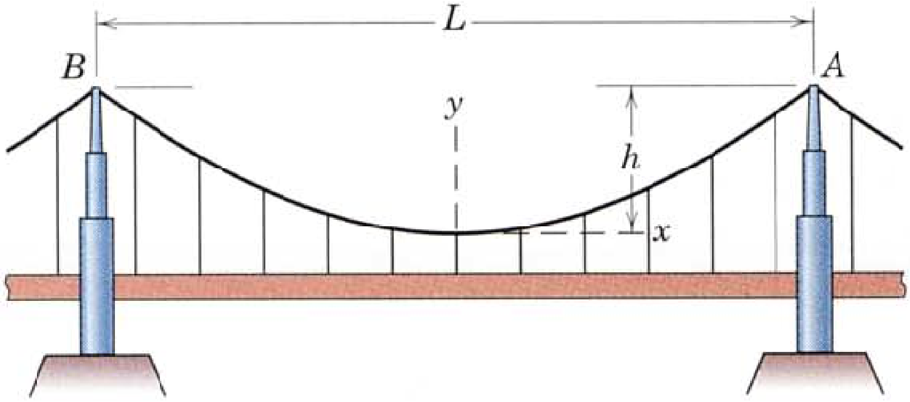
$$a = \frac{l_B^2}{2h_B} \tag{33}$$

Approximate solution

$$s_B = l_B\left[1 + \frac{2}{3}\left(\frac{h_B}{l_B}\right)^2 - \frac{2}{5}\left(\frac{h_B}{l_B}\right)^4 + \cdots\right] \tag{34}$$

Since $h_A > h_B$, the absolute maximum tension force in the cable will naturally occur at end A, since this side of the cable supports the greater proportion of the load.

Symmetric case



$s_A = s_B, \quad l_A = l_B, \quad h_A = h_B$

total span $L = 2l_A$, total sag $h = h_A$

In this case we get

$$T_{max} = \frac{wL}{2}\sqrt{1 + (L/4h)^2} \tag{35}$$

$$s = 2h\left[\sqrt{1+b^2} + b^2 \ln\left(\frac{L}{2} + \frac{L}{2}\sqrt{1+b^2}\right) - b^2 \ln\left(\frac{L^2}{8h}\right)\right] \tag{36}$$

where

$$b = \frac{L}{4h} \tag{37}$$

Approximate solution:

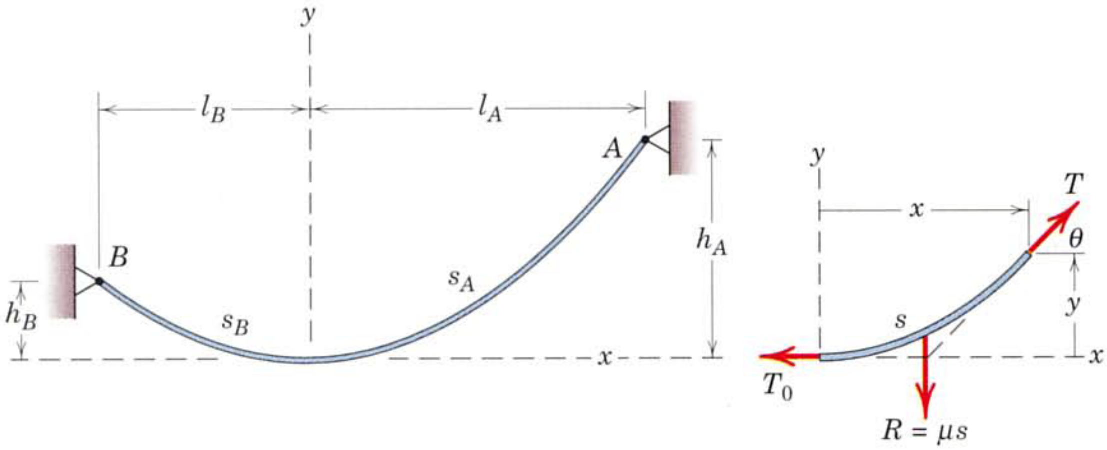
$$s = L\left[1 + \frac{8}{3}\left(\frac{h}{L}\right)^2 - \frac{32}{5}\left(\frac{h}{L}\right)^4 + \cdots\right] \tag{38}$$

This series converges for all values of $h/L < 1/4$. In most cases $h \ll L/4$.

→ The first three terms of series (38) give a sufficiently accurate approximation.

Catenary Cable

Consider cable weight only



$wx \rightarrow \mu s; \quad w\mathrm{d}x \rightarrow \mu\mathrm{d}s$

where μ is the weight per unit length of the cable in N/m.

Eq. (10): $T_H = T\cos\theta \quad \rightarrow \quad T = \frac{T_H}{\cos\theta}$

Substituting into Eq. (9) and replacing $w\mathrm{d}x$ with $\mu\mathrm{d}s$ yields

$d(T_H \tan\theta) = \mu ds, \qquad \tan\theta = \frac{dy}{dx}$

$$\rightarrow \qquad d\left(T_H \frac{dy}{dx}\right) = \mu ds \tag{39}$$

Differentiation with respect to x yield

$$\frac{d^2y}{dx^2} = \frac{\mu}{T_H}\frac{ds}{dx} \tag{40}$$

With $ds = \sqrt{(dx)^2 + (dy)^2}$, we get

$$\frac{d^2y}{dx^2} = \frac{\mu}{T_H}\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \tag{41}$$

Substitution: $p = \frac{dy}{dx} \quad \rightarrow \quad \frac{dp}{dx} = \frac{d^2y}{dx^2}$ (42)

$$\rightarrow \qquad \frac{dp}{\sqrt{1+p^2}} = \frac{dx}{c} \qquad (43)$$

where $c = \frac{T_H}{\mu}$

Substituting $p = sinhu$, $dp = coshudu$, $u = arsinhp$, gives

$$\frac{dx}{c} = \frac{coshu}{\sqrt{1+sinhu}} du \qquad (44)$$

Integrating leads to

$$\frac{x}{c} = u + C_1 \qquad (45)$$

Boundary condition:

At $x = 0$, $\frac{dy}{dx} = p = 0 \rightarrow \sinh(0) = 0 \rightarrow C_1 = 0$

So that

$$u = \frac{x}{c} = arsinhp \qquad (46)$$

or

$$p = \frac{dy}{dx} = sinh\frac{x}{c} \qquad (47)$$

which leads to

$$dy = sinh\frac{x}{c}dx \qquad (48)$$

Integrating yields

$$y = ccosh\frac{x}{c} + C_2 \qquad (49)$$

Boundary condition:

At $x = 0, y = 0$ $\rightarrow$ $C_2 = - c$

Thus, we obtain the equation of the curve formed by the cable

$$y = \frac{T_H}{\mu}\left(\cosh\frac{\mu x}{T_H} - 1\right) \tag{50}$$

Cable length

From the fee-body diagram shown in the figure we see that

$$\frac{dy}{dx} = \tan\theta = \frac{\mu s}{T_H} \rightarrow s = \frac{T_H}{\mu}\frac{dy}{dx}$$

Using Eq. (47), we get then

$$s = \frac{T_H}{\mu}\sinh\frac{\mu x}{T_H} \tag{51}$$

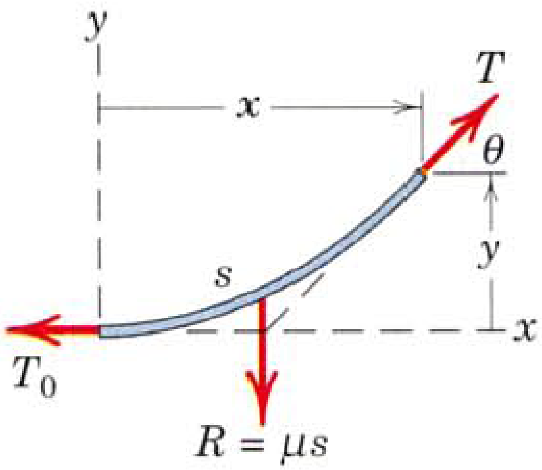
Where the unknown minimum tension force T_H may be obtained from Eq. (50) by using the boundary condition $y = h_A$ at $x = l_A$.

Tension force

From the figure, we get

$$T^2 = \mu^2 s^2 + T_H^2 \tag{52}$$

Substituting Eq. (51) into Eq. (52) leads to



$$T^2 = T_H^2\left(1 + sinh^2\frac{\mu x}{T_H}\right) = T_H^2 cosh^2\frac{\mu x}{T_H} \tag{53}$$

or

$$T = T_H cosh\frac{\mu x}{T_H} \tag{54}$$

With equation (50) we get

$$T = T_H + \mu y \tag{55}$$

The solution of catenary problems where the sag-to-span ratio is small may be approximated by the relations developed for the parabolic cable. A small sag-to-span ratio means a tight cable, and the uniform distribution of weight along the cable is not very different from the same load intensity distributed uniformly along the horizontal.