Amplification Factors of Beams with General Boundary Conditions Due to a Moving Constant Load

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Abstract: In this paper the amplification factors of an Euler-Bernoulli beam with different boundary conditions due to a moving constant load are studied. The boundary conditions considered are: pinned-pinned, fixed-fixed, pinned-fixed, and fixed-free. The effects of the beam damping and the direction of motion of the moving loads are investigated. The obtained amplification factors are given in graphical form dependent on a moving speed parameter. Also, in addition to the amplification factors of the total response of the considered beams, the amplification factors of the forced response are separately considered. The present results are compared with published results where applicable.

Keywords: Amplification factor; Beams; Moving load

ابوهلال ، محمد2007 معاملات التضخيم للجائز ذي المساند الجانبية المختلفة نتيجة مرور حمل ثابت ، قسم الهندسة الميكانيكية والصناعية، جامعة العلوم التطبيقية الخاصة، عمان 11931 الاردن. ، **المجلة الاردنية للعلوم التطبيقية**، *9(1)*، *1*-

ملخص: يعالج هذا البحث معامل التضخيم لجائز اويلر -برنولي جراء مرور أحمال ثابتة فوقه لشروط حدود مختلفة . وشروط الحدود التي تم مراعاتها هي: الإسناد المفصلي والإسناد الثابت والنهاية الحرة للجائز. وقد تم دراسة تأثير التخميد وكذلك تأثير اتجاه حركة الأحمال على معامل التضخيم . وقد تم عرض معاملات التضخيم التي تم الحصول عليها بصورة بيانيه كاقترانات من معامل سرعة حركة الأحمال المتحركة. إضافة لمعامل التضخيم الكلي فقد تم دراسة معامل التضخيم الناتج عن الحركة القصرية على انفراد. واخيرا فقد تم مقارنة النتائج التي تم الحصول عليها مع نتائج منشورة حيثما أمكن ذلك.

Introduction

Vibration of beams due to moving loads is a field of interest in mechanical, industrial and civil engineering. Vibrations of this kind occur in runways, railways, bridges, beam subjected to pressure waves and piping systems subjected to two-phase flow. The moving loads may be roughly divided into three groups: moving oscillator, moving mass, and moving force. Vibrations of beams due to moving oscillators are studied in [1-5], vibrations of beams due to a moving mass are investigated in [6-9], and vibration of beams due to a concentrated constant load is explored in [1, 10-14].

Frýba [1] studied the vibrations of beams due to a moving arbitrary force. He considered the effects of beam damping, boundary conditions, and the speed of the moving load. In his work, the amplification factor of the total deflection for only a simply supported beam is given in graphical form. He found that the maximum amplification factor is associated with speeds α= 0.5 to 0.7. Yang et al. [16] investigated the vibration of simply supported beams subjected to the passage of high speed trains. They modeled the train as the composition of two subsystems of wheel loads of constant intervals, with one consisting of all the front wheel assemblies and the other the rear assemblies. In their work, they considered the dynamic impact factor, which is defined as the difference between the maximum dynamic and maximum static responses of deflection of the beam divided by the maximum static deflection. Plots of the impact factor of a simply supported beam against a speed parameter are included in their paper. The effect of damping is considered for one value of damping ratio of 2.5%. Rao [13] studied the dynamic response of an undamped Euler-Bernoulli simply supported beam under moving loads. In the work, the normalized maximum dynamic deflection of the beam for each moving load velocity in the range 1 to 100 m/s is given in graphical form at two different positions of the beam, namely at x=0.25 L and x= 0.5 L. Thambiratnam and Zhuge [11] developed a simple procedure based on the finite element method for treating the dynamic analysis of beams on an elastic foundation subjected to moving point loads, where the foundation has been modeled by springs of variable stiffness. The effect of the speed of the moving load, the foundation stiffness and the length of the beam on the response of the beam have been studied and dynamic amplifications of deflection and stress have been evaluated. Based on the Lagrangian approach Cheung et al. [3] analyzed the vibration of a multi-span non-uniform bridge subjected to a moving vehicle by using modified beam vibration functions as the assumed modes.

The vehicle is modeled as a two-degree-of-freedom system. Obtained results are presented in form of dynamic amplification factors and compared with published results where applicable. Savin [17] derived analytic expressions of the dynamic amplification factor and the characteristic response spectrum for weakly damped beams with various boundary conditions subjected to point loads moving at constant speeds. The obtained coefficients are given as functions of the ratio of the span length to the loads wavelength, and the loads wavelength respectively.

Pesterev et al. [18] developed simple tools for finding the maximum deflection of a beam for any given velocity of the travelling force. It is shown that, for given boundary conditions, there exits a unique response-velocity dependence function. They suggested a technique to determine this function, which is based on the assumption that the maximum beam response can be adequately approximated by means of the first mode. Also, the maximum response function is calculated analytically for a simply supported beam and constructed numerically for a clamped-clamped beam. Furthermore, they investigated the effect of the higher modes on the maximum response and they constructed the relative error of the one-mode approximation for a simply supported beam. The effect of beam damping is not considered in their study.

In this paper, the total and the forced amplification factors of an Euler-Bernoulli beam with general boundary conditions subject to a moving constant force are treated. The four classical boundary conditions considered are pinned-pinned, fixed-fixed, pinned-fixed, and fixed-free. The calculated amplification factors are given in graphical form dependent on a speed parameter, since analytical expressions of the amplification factors seems to be very complicated. In the calculation, only the first term of a series solution is used and only the response of the beams during the force traverse it is considered. The free vibrations of the beams which occur after the force leaves it are not considered. The effects of the beam damping and the direction of motion of the moving force are considered. The results obtained are compared with published results where applicable.

Analytical Formulation

The transverse vibration of a uniform elastic Euler-Bernoulli beam subject to a constant load P_0 traversing the beam with a constant velocity c is governed by the partial differential equation

$$EIv'''' + \mu\ddot{v} + r_a\dot{v} + r_i\dot{v}'''' = P_0\delta(x - ct)$$

(1) where EI, μ, r_a, and r_i are the bending rigidity of the beam, the mass per unit length of the beam, the coefficient of external damping of the beam, and the coefficient of internal damping of the beam, respectively. $v(x,t)$ denotes the transverse deflection of the beam at position x and time t and $\delta(.)$ denotes the Dirac delta function. A prime denotes differentiation with respect to position x and a dot denotes differentiation with respect to time t. The external and internal damping of the beam are assumed to be proportional to the mass and stiffness of the beam respectively, i.e.;

$$r_a = \gamma_1\mu \qquad r_i = \gamma_2 EI\,,$$

(2)

where γ_1 and γ_2 are proportionality constants.

The solution of equation (1) can be represented in a series form in terms of the eigenfunctions of the beam as

$$v(x,t) = \sum_{n=1}^{\infty} X_n(x)y_n(t)$$

(3)

where $y_n(t)$ is the nth generalized deflection of the beam and $X_n(x)$ is the nth eigenfunction of the beam given as

$$X_n(x) = \sin\kappa_n x + A_n\cos\kappa_n x + B_n\sinh\kappa_n x + C_n\cosh\kappa_n x$$

(4)

where κ_n, A_n, B_n, C_n are unknown constants and can be determined from the boundary conditions of the beam. Substituting equations (2) and (3) into equation (1) and then multiplying by $X_k(x)$, and integrating with respect to x between 0 and L yields

$$\sum_{n=1}^{\infty}\int_0^L\left[(y_n + \gamma_2\dot{y}_n)EIX_n''''X_k + (\ddot{y}_n + \gamma_1\dot{y}_n)\mu X_nX_k\right]\mathrm{d}\,x = \int_0^L P_0X_k\delta(x - ct)\,\mathrm{d}\,x$$

(5)

Considering the orthogonality conditions

$$\int_0^L X_nX_k\,dx = 0 \quad n \neq k$$

(6)

and the generalized stiffness of the nth mode of the beam

$$k_n = \int_0^L EIX_n''''X_n\,dx$$

(7)

as well as the generalized mass of the beam associated with the nth mode [15]

$$m_n = \int_0^L \mu X_n^2(x)\,dx$$

(8)

yields the differential equation of the nth mode of the generalized deflection:

$$\ddot{y}_n(t) + 2\omega_n\zeta_n\dot{y}_n(t) + \omega_n^2 y_n(t) = Q_n(t)$$

(9)

where

$$\omega_n = \sqrt{k_n / m_n} = \kappa_n^2 \sqrt{EI/\mu} \quad (10)$$

is the natural circular frequency of the *n*th mode,

$$\zeta_n = \frac{\gamma_1 + \gamma_2 \omega_n^2}{2\omega_n} \qquad (11)$$

is the damping ratio of the *n*th mode, and

$$Q_n(t) = \frac{P_0}{m_n} \int_0^L X_n(x) \delta(x - ct) dx = \frac{P_0}{m_n} X_n(ct) \qquad (12)$$

is the generalized force associated with the *n*th mode of the beam.
Assuming the beam is originally at rest, i.e.:

$$v(x,0) = 0 ; \qquad \frac{\partial v(x,0)}{\partial t} = 0 , \qquad (13)$$

then the solution of equation (9) can be given as

$$y_n(t) = \int_0^t h_n(t - \tau) Q_n(\tau) d\tau \qquad (14)$$

where $h_n(t)$ is the impulse response function defined for $0 \le \zeta_n < 1$ as

$$h_n(t) = \begin{cases} \dfrac{1}{\omega_{dn}} e^{-\zeta_n \omega_n t} \sin \omega_{dn} t & t \ge 0 \\ 0 & t < 0 \end{cases} \qquad (15)$$

where

$$\omega_{dn} = \omega_n \sqrt{1 - \zeta_n^2} \qquad (16)$$

is the damped circular frequency of the *n*th mode of the beam.
Substituting equations (4), (12) and (15) into equation (14), carrying out the integration and substituting the result into equation (3) yields the total deflection of the beam for any given speed in the form

$$v(x,t) = v_h(x,t) + v_p(x,t) \qquad (17)$$

where $v_h(x,t)$ is the free vibration part of the deflection during the load traversing the beam given as

$$v_h(x,t) = \sum_{n=1}^{\infty} X_n(x) y_{hn}(t) \qquad (18)$$

and $v_p(x,t)$ is the forced vibration part of the deflection given as

$$v_p(x,t) = \sum_{n=1}^{\infty} X_n(x) y_{pn}(t) \qquad (19)$$

with

$$y_{hn} = F_n \left\{ \frac{\zeta_n \omega_n + q_{6n} A_n}{q_{1n}} - \frac{\zeta_n \omega_n + q_{5n} A_n}{q_{2n}} - \frac{q_{9n} \omega_{dn}}{q_{4n}} + \frac{q_{10n} \omega_{dn}}{q_{3n}} \right\} e^{-\zeta_n \omega_n t} \cos \omega_{dn} t$$

$$+F_n\left\{\frac{q_{6n}-\zeta_n\omega_n A_n}{q_{1n}}+\frac{q_{5n}-\zeta_n\omega_n A_n}{q_{2n}}-\frac{q_{7n}q_{9n}}{q_{4n}}+\frac{q_{8n}q_{10n}}{q_{3n}}\right\}e^{-\zeta_n\omega_n t}\sin\omega_{dn}t$$

(20)

and

$$y_{pn}(t)=F_n\left[\frac{\zeta_n\omega_n+q_{5n}A_n}{q_{2n}}-\frac{\zeta_n\omega_n+q_{6n}A_n}{q_{1n}}\right]\cos\kappa_n ct+\frac{F_n\omega_{dn}q_{9n}}{q_{4n}}e^{\kappa_n ct}$$

$$+F_n\left[\frac{q_{5n}-\zeta_n\omega_n A_n}{q_{2n}}-\frac{q_{6n}-\zeta_n\omega_n A_n}{q_{1n}}\right]\sin\kappa_n ct-\frac{F_n\omega_{dn}q_{10n}}{q_{3n}}e^{-\kappa_n ct}$$

(21)

where

$$F_n=\frac{P_0}{2m_n\omega_{dn}}$$

$$q_{1n}=(\zeta_n\omega_n)^2+(\kappa_n c-\omega_{dn})^2 \qquad q_{2n}=(\zeta_n\omega_n)^2+(\kappa_n c+\omega_{dn})^2$$

$$q_{3n}=\omega_{dn}^2+(\zeta_n\omega_n-\kappa_n c)^2 \qquad q_{4n}=\omega_{dn}^2+(\zeta_n\omega_n+\kappa_n c)^2$$

$$q_{5n}=\kappa_n c+\omega_{dn} \qquad q_{6n}=\kappa_n c-\omega_{dn}$$

$$q_{7n}=\zeta_n\omega_n+\kappa_n c \qquad q_{8n}=\zeta_n\omega_n-\kappa_n c$$

$$q_{9n}=B_n+C_n \qquad q_{10n}=B_n-C_n \qquad \text{(22a-k)}$$

Approach

Equations (17) and (19) are used to determine the amplification factors for the studied beams.

For each of the considered beams, graphs are plotted of the normalized deflections

$$\bar{v}=\frac{v(x_{\max},s)}{v_0} \qquad (23)$$

and

$$\bar{v}_p=\frac{v_p(x_{\max},s)}{v_0} \qquad (24)$$

versus the dimensionless time s for various values of a speed parameter α, where v_0 and x_{max} are the maximum static deflection of the beam and the position at which v_0 occurs respectively [15]. The dimensionless time is defined as

$$s=\frac{t}{t_e} \qquad (25)$$

where

$$t_e=\frac{L}{c} \qquad (26)$$

is the time, the force needs to cross the beam. Thus when s=0 (t=0) the force is at the left-hand side of the beam, i.e. x=0, and when s=1 (t=t_e) the force is at the right-hand side of the beam (x=L). The dimensionless speed parameter α is defined as

$$\alpha=\frac{c}{c_{cr}}, \qquad (27)$$

with the critical speed c_{cr} defined as [1]

$$c_{cr} = 2 f_1 L = \frac{2L}{\tau_1} \qquad (28)$$

where f_1 and τ_1 are the first natural frequency and the first natural period of the considered beam, respectively. That is to say, the speed parameter α is defined in such a way that, when α equals unity, the vehicle traversing time $t_e=L/c$ equals half the fundamental period of the beam [3]. An arbitrary point on the beam makes during the force traversing the beam a number of free oscillations given as

$$N = \frac{t_e}{\tau_1} = \frac{1}{2\alpha} \quad . \qquad (29)$$

To be specific, forty graphs of each of the normalized deflections $\bar{v}(x_{\max}, s)$ and $\bar{v}_p(x_{\max}, s)$ are plotted for each considered beam and damping ratios ζ=0, 0.1, 0.2 and 0.5 for different values of α between 0 and 2; except for $\bar{v}_p$ of the cantilevered beams where the values of α are extended to 3. The used values of α are not equidistant. In the critical regions a smaller interval $\Delta\alpha$ is selected in order to obtain representative curves. After that, from each of the plotted graphs, the maximum value of $\bar{v}$ or of $\bar{v}_p$ is obtained. These values are then plotted versus the corresponding values of α in a graph and spline-interpolated in order to get smooth curves. The outcoming curves represent finally the sought amplification factors of the studied beams.

Results and Discussion

Figures 1 and 2 show amplification factors for the studied beams. In Figures 1, amplification factors Φ_T of the total deflection (total amplification factor) are plotted whereas Figure 2 shows amplification factors of the forced vibration part of the deflection (forced amplification factor). The total amplification factor Φ_T is defined as the maximum total deflection of a beam calculated at an arbitrary position x^* divided by the static deflection v_{st} calculated at the same position x^*, i.e.:

$$\Phi_T = \frac{Max\{v(x^*, s)\}}{v_{st}(x^*)} \qquad (30)$$

The forced amplification factor Φ_F is defined as the maximum of the forced vibration part of the deflection of a beam calculated at an arbitrary position x^* divided by the static deflection v_{st} calculated at the same position x^*, i.e.:

$$\Phi_F = \frac{Max\{v(x^*, s)\}}{v_{st}(x^*)} \qquad (31)$$

Both amplification factors are calculated in this work only for the time interval $t \in [0, t_e]$. The amplification factors are determined here at the position x_{max}, where the maximum static deflection of the beam occurs, i.e.:

$$\Phi_T = \frac{Max\{v(x_{\max}, s)\}}{v_0} \quad (32)$$

and

$$\Phi_F = \frac{Max\{v(x_{\max}, s)\}}{v_0} \quad (33)$$

Figure 1-a shows the total amplification factor Φ_T for a pinned-pinned beam plotted against the speed parameter α for different values of damping ζ. From the figure, the following characteristics can be noted:

1. For the undamped beam (ζ=0), the values of Φ_T remain limited, even for α=1, and take positive values smaller than 1.75. Also resonance phenomenon does not occur.
2. In general, increasing the damping leads to decreasing the amplification factor for all values of α.
3. In the static case α=0, the value of Φ_T =1, this means that the dynamic deflection is equal to the static deflection.
4. By holding the damping constant and increasing the values of α, the amplification factor increases until it reaches an absolute maximum and then decreases monotonically. These maxima are given for different values of damping ratios in Table 1.
5. By increasing the damping, the absolute maximum of the amplification factor occurs at a smaller speed factor.
6. For the used damping ratios $0 \leq \zeta \leq 0.5$, the maximum amplification factor is associated with speed parameter α=0.418 to 0.617 and its value lies between 1.064 and 1.743.

The absolute maxima $\Phi_{T,max}$ of the total amplification factors and their corresponding speed parameters α are given in Table 1 for various damping ratios and different beams.

Comparison of the curves plotted in Figure 1-a for ζ =0 and ζ=0.1 with those published in [1] Figure 1.3 yields excellent agreement except for small values of α<0.25. But comparing the curve for ζ=0 with those published in [18], Figure 1, $\Phi_1(\beta_1)$, and in [17], Figure 2, solid curve yields excellent agreement. Also in Figure 1 published in [18] it is evident that at speeds $\alpha \leq 1$ the maximum total amplification factor is produced already during the force passage. The swinging behavior of the amplification factor at speeds α<0.25 is explained by the free vibration part of the deflection (equation (18)).

Figures 1-b-d show total amplification factors for fixed-fixed, pinned-fixed, and fixed-pinned beams, respectively. The curves in these figures show similar behavior to those in Figure 1-a. However, at α=0, the amplification factors deviate slightly from 1 due to the one-term approximation. The values $\Phi_{T,max}$ and their corresponding speeds are given in Table 1. Comparison of Φ_T for an undamped fixed-fixed beam as plotted in Figure 1-b with Figure 5, Φ_{1cc} published in [18] yields good agreement.

Figure 1-e shows total amplification factors for a fixed-free beam for different damping ratios. Qualitatively, the curves show similar behavior to those of a pinned-pinned beam. However, the values of Φ_T for the fixed-free beam are very smaller. Furthermore the swinging behavior of Φ_T for this beam is stretched to a wider α-range. Also the amplification factor for ζ=0.5 decreases monotonically and shows no maximum.

Figure 1-f shows total amplification factors for a free-fixed beam. From the figure it is clear to see that the amplification factor for this beam shows a very different behavior to the amplification factors for the previously discussed beams. The curves in the figure decrease monotonically by increasing the values of α. Also

these curves may be approximated with very good accuracy as cubic polynomial of the form

$$\Phi_T(\alpha) = a_3\alpha^3 + a_2\alpha^2 + a_1\alpha + a_0 \qquad (34)$$

where a_1 and a_3 are negative and a_0 and a_2 are positive real-valued constants. Furthermore, increasing the damping leads to a decrease in the values of Φ_T for all values of α.

Figure 2 shows forced amplification factors versus the speed parameter α for the studied beams for different values of damping. The curves of the undamped beams show resonance at the speed parameter $\alpha_R = \pi/\lambda_i$, where λ_i is the first eigenvalue of the considered beam.

Figure 2-a shows the forced amplification factor for a pinned-pinned beam. From the figure it is clear to see that the forced amplification factor Φ_F for the pinned-pinned beam behaves similar to the amplification factor of a system with single degree of freedom excited by a harmonic force acting on its mass. The maxima $\Phi_{F,max}$ of the forced amplification factors and their corresponding speed parameters are given in Table 2 for various damping ratios and different beams.

Figures 2-b-d show amplification factors for fixed-fixed, pinned-fixed, and fixed-pinned beams, respectively. Qualitatively, the curves in these figures behave similar to those in Figure 2-a. However, it is to observe from Figures 2-b and 2-d that increasing the damping may lead to increasing the values of Φ_F for the fixed-fixed, and pinned-fixed beams at supercritical speeds $\alpha > \alpha_R$.

Figure 2-e shows forced amplification factors for a fixed-free beam. The amplification factor of the undamped beam shows similar behavior to that of a pinned-pinned beam. From the figure, it is observed that increasing the damping leads to a decrease in the amplification factor for all values of α. For small values of damping and by increasing the values of α, the amplification factor increases until it reaches a maximum and then decreases. For higher damping with increasing the values of α, the amplification factor decreases monotonically down to a sharp bend then increases to a maximum and finally decreases. Also, each of the amplification factors of the damped beam show two bends, the first and sharper one occurs in the subcritical region whereas the other one in the supercritical region. In general, the amplification factors for the fixed-free beam show smaller values than those for the previously discussed beams.

Figure 2-f shows forced amplification factors for a free-fixed beam. From the figure it is to observe that by increasing the values of α, the amplification factor increases until it reaches a maximum and then decreases. Also increasing the damping leads to increasing the values of the amplification factor for values $\alpha \le 1$ and higher values of α in the supercritical region. The damping has a decreasing effect especially in the resonance region.

From Figures 2-e and 2-f it could be observed that in general, the cantilevered beams have smaller amplification factors in the resonance region than the previously discussed beams.

In order to show the effect of direction of motion of the moving load on the amplification factor of the cantilevered beams, we compare Figures 2-e and 2-f with each other. From these figures it is evident to see that for the same damping ratio, the amplification factor for a free-fixed beam is essentially greater than the amplification factor of a fixed-free beam for all values of α. Also the same effect could be observed by comparing the amplification factors for the pinned-fixed, and fixed-pinned beams together. However, the difference in their values is smaller than by the cantilevered beams.

Conclusions

The problem of determining the amplification factors for a beam with general boundary conditions subject to a moving constant force has been investigated. It has been found that there exists a unique function describing the dependence of the amplification factor of a beam with general boundary conditions on the speed of the moving force. The total amplification factor and the forced amplification factor for pinned-pinned, fixed-fixed, pinned-fixed, fixed-free beams are given in graphical form. It is found that the total amplification factor for the studied beams, except the free-fixed beam, increases until it reaches an absolute maximum and then decreases monotonically by increasing the values of the speed parameter α. The absolute maxima $\Phi_{T,max}$ are smaller than 1.82 and occur at speed parameters $0.32<\alpha<0.62$. The total amplification factor for a free-fixed beam decreases monotonically by increasing α the values of and may be approximated as a cubic polynomial. Qualitatively, the forced amplification factors Φ_F for the studied beams behave similar to the amplification factor of a system with single degree of freedom excited by a harmonic force acting on its mass. Furthermore, the amplification factors show resonance at the speed parameter $\alpha_R = \pi / \lambda_i$, where λ_i is the first eigenvalue of the considered beam. Also the effects of damping and the direction of motion of the travelling force are examined. The obtained results are general and can be used independent of the geometrical (L, A, I) and material (E, μ) properties of the beams since in presenting the results dimensionless quantities are used.

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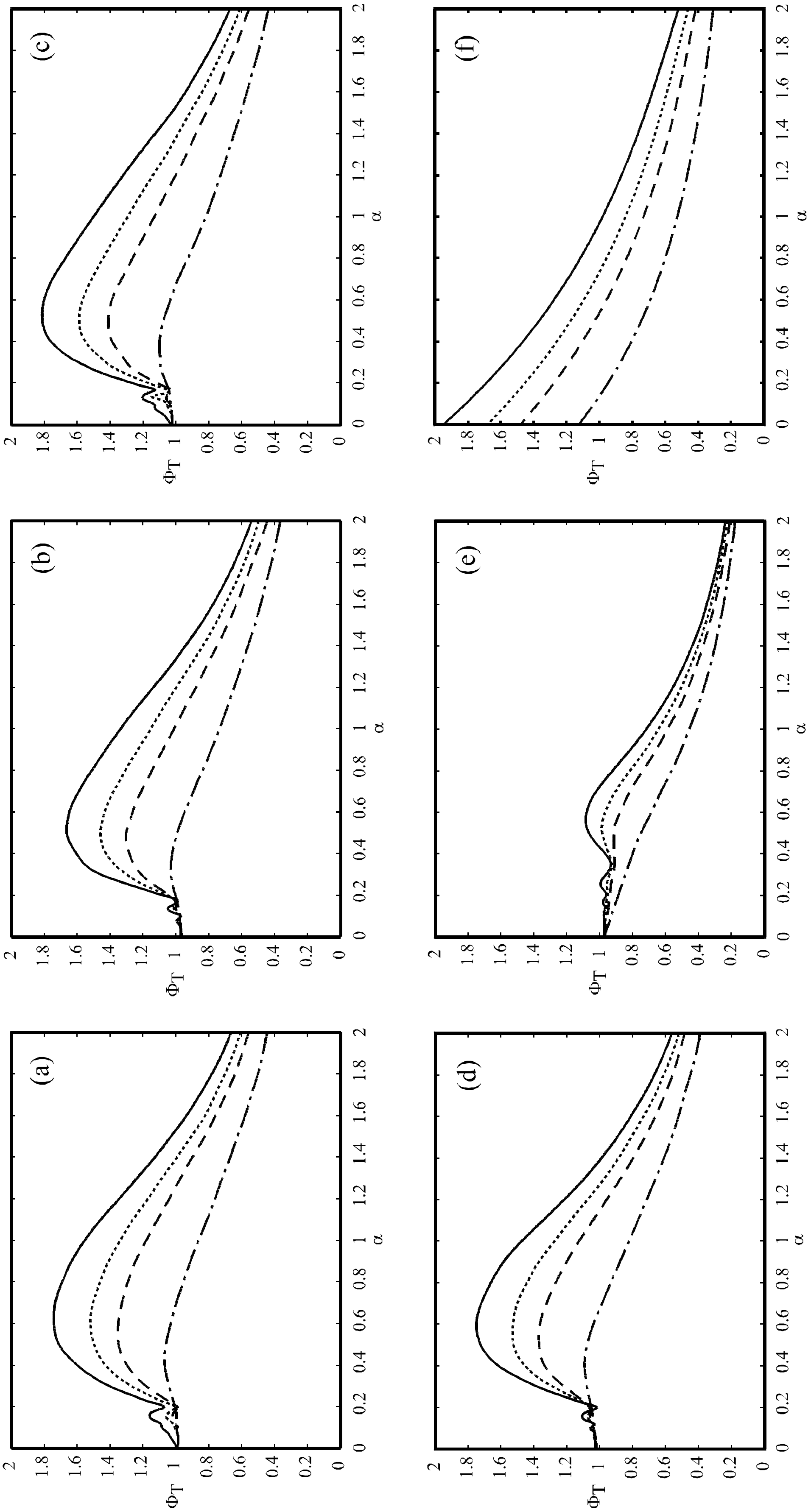
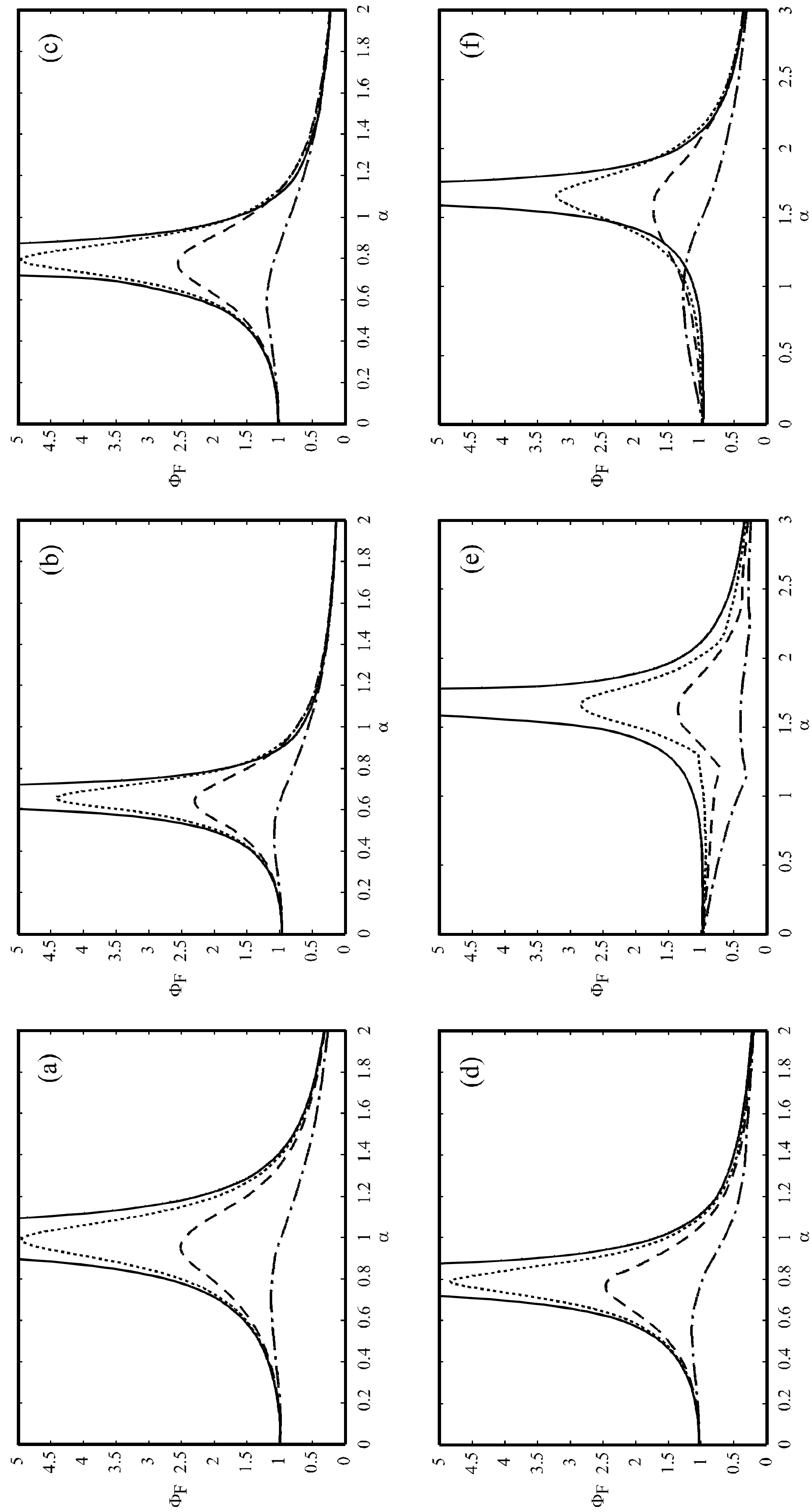
Table 1. The absolute maxima $\Phi_{T,max}$ of the total amplification factor and their corresponding speed parameters α for various damping ratios and different beams.

	pinned-pinned				fixed-fixed			
ζ	0	0.1	0.2	0.5	0	0.1	0.2	0.5
α	0.617	0.592	0.560	0.418	0.520	0.495	0.465	0.332
$\Phi_{T,\max}$	1.743	1.520	1.354	1.064	1.664	1.456	1.301	1.032
	pinned-fixed				fixed-pinned			
ζ	0	0.1	0.2	0.5	0	0.1	0.2	0.5
α	0.525	0.503	0.476	0.355	0.595	0.565	0.530	0.380
$\Phi_{T,\max}$	1.815	1.583	1.410	1.106	1.751	1.533	1.370	1.089
	fixed-free				free-fixed			
ζ	0	0.1	0.2	0.5	0	0.1	0.2	0.5
α	0.56	0.52	0.46	-	-	-	-	-
$\Phi_{T,\max}$	1.086	0.984	0.915	-	-	-	-	-

Table 2. The absolute maxima $\Phi_{F,max}$ of the forced amplification factor and their corresponding speed parameters α for various damping ratios and different beams.

	pinned-pinned				fixed-fixed			
ζ	0	0.1	0.2	0.5	0	0.1	0.2	0.5
α	1	0.99	0.959	0.707	0.664	0.659	0.642	0.469
$\Phi_{F,\max}$	∞	4.953	2.515	1.138	∞	4.414	2.288	1.081
	pinned-fixed				fixed-pinned			
ζ	0	0.1	0.2	0.5	0	0.1	0.2	0.5
α	0.8	0.794	0.774	0.587	0.8	0.792	0.766	0.541
$\Phi_{F,\max}$	∞	4.951	2.560	1.186	∞	4.822	2.453	1.137
	fixed-free				free-fixed			
ζ	0	0.1	0.2	0.5	0	0.1	0.2	0.5
α	1.675	1.661	1.627	1.541	1.675	1.655	1.571	0.936
$\Phi_{F,\max}$	∞	2.856	1.351	0.399	∞	3.197	1.735	1.272


(a)
(b)
(c)
(d)
(e)
(f)
α
Φ_T
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2


(a)
(b)
(c)
(d)
(e)
(f)
Φ_F
α