

A SEMIRELATIVISTIC TREATMENT OF SPINLESS PARTICLES SUBJECT TO THE YUKAWA POTENTIAL WITH ARBITRARY ANGULAR MOMENTA

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We obtain analytical solutions of the two-body spinless Salpeter (SS) equation with the Yukawa potential within the conventional approximation scheme to the centrifugal term for any l -state. The semirelativistic bound-state energy spectra and the corresponding normalized wave functions are calculated by means of the Nikiforov–Uvarov (NU) method. We also obtain the numerical energy spectrum of the SS equation without any approximation to centrifugal term for the same potential and compare them with the approximated numerical ones obtained from the analytical expressions. It is found that the exact numerical results are in good agreement with the approximated ones for the lower energy states. Special cases are treated like the nonrelativistic limit and the solution for the Coulomb problem.

Keywords: Spinless Salpeter equation; Yukawa potential; Coulomb potential; Schrödinger equation; Nikiforov–Uvarov method.

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1. Introduction

The Bethe–Salpeter equation,¹ named after Hans Bethe and Edwin Salpeter, describes the bound-states of a two-body (particles) quantum field theoretical system in a relativistic covariant formalism. The equation was actually first published in 1950 at the end of a paper by Yoichiro Nambu but without derivation.²

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Due to its generality and its application in many branches of theoretical physics, the Bethe–Salpeter equation appears in many different forms.^{3–10} Different aspects of this equation have been elegantly studied mainly by Lucha *et al.*^{11–24} and some other authors where they have worked on many interesting approaches to deal with its nonlocal Hamiltonian.^{25–32}

The Yukawa potential or static screening Coulomb potential (SSCP) is given by

$$V(r) = -V_0 \frac{e^{-ar}}{r}, \quad (1)$$

where $V_0 = \alpha Z$, $\alpha = 1/137.037$ is the fine-structure constant and Z is the atomic number and is the screening parameter. This potential is often used to compute bound-state normalizations and energy levels of neutral atoms^{33–38} which have been studied over the past years. It is known that SSCP yields reasonable results only for the innermost states when Z is large. However, for the outermost and middle atomic states, it gives rather poor results. The bound-state energies of the SSCP with $Z = 1$ have been studied in the light of the shifted large- N method.³⁹ Ikhdair and Sever investigated energy levels of neutral atoms by applying an alternative perturbative scheme in solving the Schrödinger equation for the Yukawa potential model with a modified screening parameter.³⁸ They studied bound-states of the Hellmann potential, which represents the superposition of the attractive Coulomb potential $-a/r$ and the Yukawa potential $b \exp(-\delta r)/r$ of arbitrary strength b and the screening parameter δ .⁴⁰ They also considered the bound-states of the exponential cosine-screened Coulomb potential⁴¹ and a more general exponential screened Coulomb (MGESC) potential.⁴²

Karakoc and Boztosun applied the asymptotic iteration method to solve the radial Schrödinger equation for the Yukawa type potentials.⁴³ Gönül *et al.* proposed a new approximation scheme to obtain analytic expressions for the bound-state energies and wave functions of Yukawa like potentials.⁴⁴

Very recently, Ikhdair has approximately solved the relativistic Dirac equation with Yukawa potential for any spin-orbit quantum number in the presence of spin and pseudospin symmetry.⁴⁵ Liverts *et al.* used the quasi-linearization method (QLM) for solving the Schrödinger equation with Yukawa potential.⁴⁶

The aim of the present work is to study the Yukawa potential within the framework of a semirelativistic spinless Salpeter (SS) equation. First, we approximately calculate the bound-state energy eigenvalues and their corresponding un-normalized wave functions with arbitrary l -states for the short-range Yukawa potential by using a parametric generalization of the NU method.⁴⁷ This shortcut is a powerful tool in solving second-order differential equation and has proved its effectiveness in solving various potential models over the past few years. Second, we are going to compare our approximated numerical energy eigenvalues with the exact numerical ones calculated by solving the same quantum system without making approximation to the centrifugal term.

The structure of the paper is as follows. In Sec. 2, we review the SS equation and apply it to the Yukawa potential interaction to obtain the semirelativistic SS bound-state energy spectrum and their corresponding normalized wave functions for two interacting particles. In Sec. 2, we reduce our solution into the nonrelativistic limit and obtain the Schrödinger bound-states of the energy spectrum and normalized wave functions. Further, we obtain the nonrelativistic solution of the Coulomb field. Finally, in Sec. 3 we give our final comments and conclusion.

2. Spinless Salpeter Equation and Application to Yukawa Potential

The SS equation for a two-body system under a spherically symmetric potential in the center-of-mass system has the form^{25,26,30,31}

$$\left[\sum_{i=1,2} \sqrt{-\Delta + m_i^2} + V(r) - M \right] \chi(\mathbf{r}) = 0, \quad \Delta = \nabla^2, \quad (2)$$

where the kinetic energy terms involving the operation $\sqrt{-\Delta + m_i^2}$ are nonlocal operators and $\chi(\mathbf{r}) = R_{nl}(r)Y_{lm}(\theta, \phi)$, where $R_{nl}(r) = r^{-1}\psi_{nl}(r)$, stands for the semirelativistic wave function. For heavy interacting particles, the kinetic energy operators in Eq. (2) can be approximated as²⁻³²

$$\sqrt{-\Delta + m_i^2} = m_1 + m_2 - \frac{\Delta}{2\mu} - \frac{\Delta^2}{8\eta^3} - \dots, \quad (3)$$

where $\mu = m_1 m_2 / (m_1 + m_2)$ stands for the reduced mass and $\eta = \mu(m_1 m_2 / (m_1 m_2 - 3\mu^2))^{1/3}$ is an introduced useful mass parameter.^{25,26} The above Hamiltonian containing relativistic corrections up to order (v^2/c^2) and is called a generalized Breit-Fermi Hamiltonian.¹¹⁻¹⁵ Using an appropriate transformation and following the procedures explained in Ref. 30 [see Eqs. (13)–(18)], one can then rewrite the semirelativistic SS equation as

$$\left[-\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + \frac{l(l+1)\hbar^2}{2\mu r^2} + W_{nl}(r) - \frac{W_{nl}^2(r)}{2\tilde{m}} \right] \psi_{nl}(r) = 0, \quad (4)$$

where

$$W_{nl}(r) = V(r) - E_{nl}, \quad \text{and} \quad \tilde{m} = \frac{\eta^3}{\mu^2} = \frac{m_1 m_2 \mu}{m_1 m_2 - 3\mu^2}. \quad (5)$$

Now, we intend to solve the above semirelativistic SS equation with the Yukawa potential interaction (1). Thus, the insertion of Eq. (1) into Eq. (5) allows one to obtain

$$\left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + \frac{2\mu}{\hbar^2} \left(V_0 \frac{e^{-ar}}{r} + E_{nl} + \frac{V_0^2}{2\tilde{m}} \frac{e^{-2ar}}{r} + \frac{E_{nl} V_0}{\tilde{m}} \frac{e^{-ar}}{r} + \frac{E_{nl}^2}{2\tilde{m}} \right) \right] \psi_{nl}(r) = 0. \quad (6)$$

Since Eq. (6) does not admit exact analytical solution due to the presence of the strong singular centrifugal term r^{-2} , we resort to use a proper approximation to deal with this term. So, we employ the conventional approximation scheme introduced by Greene and Aldrich^{48–50}:

$$\frac{1}{r^2} \approx 4a^2 \frac{e^{-2ar}}{(1 - e^{-2ar})^2}, \quad (7a)$$

$$\frac{1}{r} \approx 2a \frac{e^{-ar}}{(1 - e^{-2ar})}, \quad (7b)$$

which is valid only for a short-range potential, i.e., $ar \ll 1$ and used only to calculate the lowest energy states as mentioned in Ref. 50. Therefore, to see the accuracy of our approximation, we plot the Yukawa potential (1) and its approximation⁴⁵

$$V(r) = -2aV_0 \frac{e^{-2ar}}{(1 - e^{-2ar})}, \quad (8)$$

with parameters $V_0 = 1.0$ and $a = 0.01 \text{ fm}^{-1}$, as shown in Fig. 1. Thus, the approximate analytic solution of the SS equation with the Yukawa potential can be obtained by inserting Eq. (7) into Eq. (6) as

$$\left[\frac{d^2}{dr^2} - 4a^2 \frac{l(l+1)e^{-2ar}}{(1 - e^{-2ar})^2} + \frac{2\mu}{\hbar^2} \left(2aV_0 \frac{e^{-2ar}}{(1 - e^{-2ar})} + E_{nl} + \frac{2a^2V_0^2}{\tilde{m}} \frac{e^{-4ar}}{(1 - e^{-2ar})^2} + \frac{2aE_{nl}V_0}{\tilde{m}} \frac{e^{-2ar}}{(1 - e^{-2ar})} + \frac{E_{nl}^2}{2\tilde{m}} \right) \right] \psi_{nl}(r) = 0. \quad (9)$$

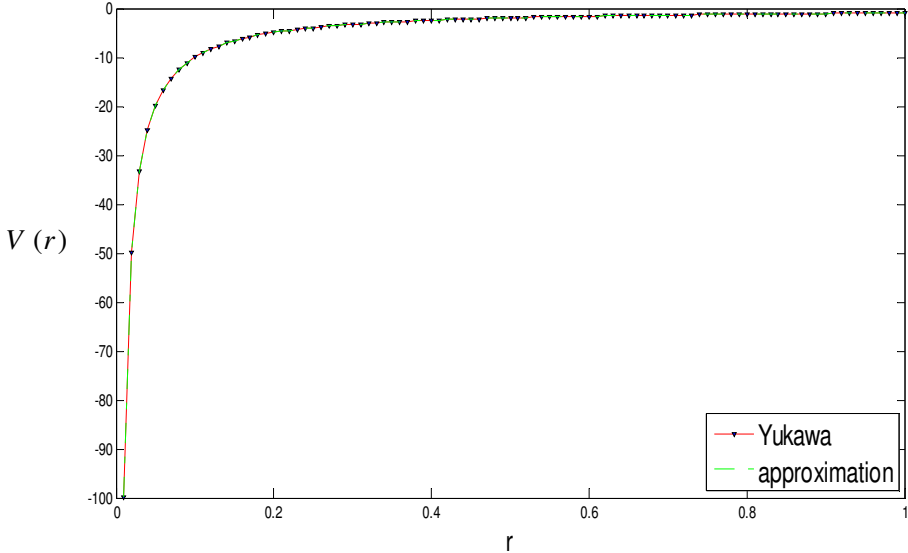


Fig. 1. (Color online) Yukawa potential (1) (red curve) and its approximation (8) (green curve).

Now, we can recast the above equation into a simpler form amenable to the NU⁴⁷ solution by making the change of variable $s = e^{-2ar}$ to obtain

$$\left[\frac{d^2}{ds^2} + \frac{1-s}{s(1-s)} \frac{d}{ds} + \frac{1}{s^2(1-s)^2} (-As^2 + Bs - C) \right] \psi_{nl}(s) = 0, \quad (10)$$

where

$$A = \frac{\mu}{\hbar^2} \left(V_0 - \frac{E_{nl}}{2a} \right) \left[\frac{1}{a} - \frac{1}{\tilde{m}} \left(V_0 - \frac{E_{nl}}{2a} \right) \right], \quad (11a)$$

$$B = -l(l+1) + \frac{\mu}{\hbar^2 a} \left[\left(V_0 - \frac{E_{nl}}{a} \right) + \frac{E_{nl}}{\tilde{m}} \left(V_0 - \frac{E_{nl}}{2a} \right) \right], \quad (11b)$$

$$C = -\frac{\mu}{\hbar^2} \frac{E_{nl}}{2a} \left(1 + \frac{E_{nl}}{2\tilde{m}} \right), \quad (11c)$$

In Appendix A, we present a shortcut to the NU method used to obtain the analytical solution of Eq. (10). Hence, comparing Eq. (10) with (A.1), we find the parametric coefficients:

$$c_1 = 1, \quad c_2 = 1, \quad c_3 = 1. \quad (12)$$

In addition, the relation (A.5) determines the rest of the coefficients as

$$\begin{aligned} c_4 &= 0, & c_5 &= -\frac{1}{2}, \\ c_6 &= \frac{1}{4} + A, & c_7 &= -B, \\ c_8 &= C, & c_9 &= \frac{1}{4} + A - B + C = -\frac{\mu V_0^2}{\hbar^2 \tilde{m}} + \left(l + \frac{1}{2} \right)^2, \\ c_{10} &= 2\sqrt{C}, & c_{11} &= \sqrt{1 + 4(A - B + C)}. \\ c_{12} &= \sqrt{C}, & c_{13} &= \frac{1}{2} \left[1 + \sqrt{1 + 4(A - B + C)} \right]. \end{aligned} \quad (13)$$

Therefore, from relation (A.10), we can obtain the binding energy equation as

$$\sqrt{\frac{\mu}{\hbar^2} \left[\frac{1}{a} \left(V_0 - \frac{E_{nl}}{2a} \right) - \frac{1}{\tilde{m}} \left(\frac{E_{nl}}{2a} - V_0 \right)^2 \right]} - \sqrt{-\frac{\mu}{\hbar^2} \frac{E_{nl}}{2a^2} \left(V_0 + \frac{E_{nl}}{2\tilde{m}} \right)} = n + v + \frac{1}{2}, \quad (14a)$$

$$v = \sqrt{\left(l + \frac{1}{2} \right)^2 - \frac{\mu}{\hbar^2} \frac{V_0^2}{\tilde{m}}}, \quad (14b)$$

where v is a new quantum number and the binding energy solution is chosen to be negative (i.e., $E_{nl} < 0$) and small satisfying $E_{nl} \ll \tilde{m}$.^{25,26} Equation (14a) appears as a complicated transcendental energy equation acquiring two solutions. However, we select the negative binding energy as a physical solution to the transcendental equation.

Table 1. Approximation and numerical binding energy spectrum of the SSE with Yukawa potential for various values of n and l quantum numbers.

		$-E_{nl}$					
		$a = 0.01 \text{ fm}^{-1}$		$a = 0.01 \text{ fm}^{-1}$		$a = 0.01 \text{ fm}^{-1}$	
n	l	Approximation	Numerical	Approximation	Numerical	Approximation	Numerical
1	0	0.5032	0.5035	0.5082	0.5084	0.5122	0.5124
2	0	0.1843	0.1912	0.1892	0.1960	0.1932	0.2000
	1	0.0709	0.0718	0.07568	0.0765	0.0796	0.0803
3	0	0.0907	0.0978	0.0956	0.1025	0.0995	0.1064
	1	0.0419	0.0416	0.0465	0.0460	0.0504	0.0499
	2	0.0258	0.0258	0.0303	0.0300	0.0341	0.0337
4	0	0.0516	0.0568	0.0563	0.0615	0.0602	0.0653
	1	0.0262	0.0266	0.0307	0.0308	0.0345	0.0346
	2	0.0167	0.0165	0.0210	0.0204	0.0248	0.0241
	3	0.0109	0.0116	0.0149	0.0153	0.0187	0.0189
5	0	0.0317	0.0355	0.0362	0.0399	0.0401	0.0437
6	0	0.0203	0.0231	0.0247	0.0273	0.0285	0.0311
7	0	0.0133	0.0154	0.0174	0.0194	0.0211	0.0231

To show the accuracy of this approximation scheme, we calculated the exact numerical energy eigenvalues of Eq. (6) without making approximation to the centrifugal term by using the finite difference method^{51–53} for arbitrary radial and orbital quantum numbers n and l for different values of the screening parameter a . The results are given in Table 1 and compared with those obtained by using Eq. (14). In the low screening region, for the (n, l) states when the screening parameter $a = 0.001 \text{ fm}^{-1}$, we found the percentage error $|\frac{E(\text{approx}) - E(\text{num})}{E(\text{num})}| \%$ for a few lowest states: 0.0454%; 3.403% and 0.866%; 6.478%, 0.936% and 1.292% for (1,0); (2,0) and (2,1); (3,0), (3,1) and (3,2), respectively. However, the percentage error increases with the increasing of the screening parameter, $a = 0.01 \text{ fm}^{-1}$, it becomes 0.269%; 3.598% and 1.238%; 7.206%, 0.596% and 0.150%. We take the following set of physical parameter values, $m_1 = 5 \text{ fm}^{-1}$, $m_2 = 5 \text{ fm}^{-1}$, $V_0 = 1$ and $a = (0.01, 0.005, -0.001) \text{ fm}^{-1}$. It is found that the numerical energy states obtained by means of Eq. (6) are in good agreement with the ones obtained by Eq. (14) for low screening regime, i.e., $a \ll 1$ since the approximation is valid only for the lowest states.⁵⁰

Now, we seek to find the corresponding wave functions, referring to Eq. (13) and relations (A.11) and (A.12) of Appendix A, we find the functions

$$\rho(s) = s^{2\varepsilon_{nl}}(1-s)^{2v}, \quad (15a)$$

$$\phi(s) = s^{2\varepsilon_{nl}}(1-s)^{\frac{1}{2}+v}, \quad (15b)$$

where

$$\varepsilon_{nl} = \sqrt{-\frac{\mu}{2\hbar^2} \frac{E_{nl}}{a^2} \left(1 + \frac{E_{nl}}{2\tilde{m}}\right)}. \quad (16)$$

Hence, relation (A.13) gives

$$y_n(s) = P_n^{(2\varepsilon_{nl}, 2v)}(1 - 2s). \quad (17)$$

By using $R_{nl}(s) = \phi(s)y_n(s)$, we get the radial SS wave functions for Yukawa potential from relation (A.14) as

$$\psi_{nl}(s) = A_{nl}s^{\varepsilon_{nl}}(1 - s)^{\frac{1}{2}+v}P_n^{(2\varepsilon_{nl}, 2v)}(1 - 2s) \quad (18)$$

and upon substituting $s = e^{-2ar}$, we obtain

$$\psi_{nl}(r) = A_{nl}(e^{-2ar})^{\varepsilon_{nl}}(1 - e^{-2ar})^{\frac{1}{2}+v}P_n^{(2\varepsilon_{nl}, 2v)}(1 - 2e^{-2ar}), \quad (19)$$

where the normalization constant A_{nl} is calculated in details in Appendix B as

$$A_{n;} = \sqrt{\frac{2a\varepsilon_{nl}n!(2n + 2v + 2\varepsilon_{nl} + 1)\Gamma(n + 2v + 2\varepsilon_{nl} + 1)}{(n + v + 1/2)\Gamma(n + 2v + 1)\Gamma(n + 2\varepsilon_{nl} + 1)}}. \quad (20)$$

The new quantum number v and the energy parameter ε_{nl} appearing in Eq. (19) are defined in Eqs. (14b) and (16), respectively. To show the accuracy of our approximation, we plot in Fig. 2 the approximated, and numerical wave functions of the SSE with the Yukawa potential for $n = 1$ and $l = 0$.

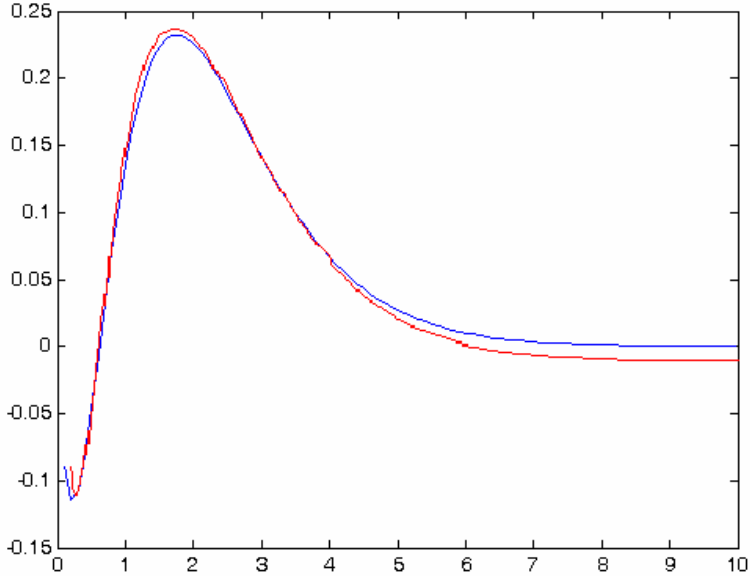


Fig. 2. (Color online) Approximation (red curve) and numerical (blue curve) wave function of the SSE with Yukawa potential for $n = 1$ and $l = 0$.

3. Special Cases

First, we consider the nonrelativistic limiting case when $W_{nl}(r) \ll \tilde{m}$, then $W_{nl}^2(r) \approx 0$, from which Eq. (4) can be recast in the form of Schrödinger equation having the energy formula and wave function obtained from Eqs. (14) and (19) as

$$E_{nl} = -a \left[\frac{\hbar^2 a}{2\mu} (n+l+1)^2 + \frac{\mu}{2\hbar^2 a} \frac{V_0^2}{(n+l+1)^2} - 2V_0 \right], \quad (21)$$

and

$$\psi_{nl}(r) = C_{nl} (e^{-2ar})^{\lambda_{nl}} (1 - e^{-2ar})^{l+1} P_n^{(2\lambda_{nl}, 2l+1)}(1 - 2e^{-2ar}), \quad \lambda_{nl} = \frac{1}{a} \sqrt{-\frac{\mu E_{nl}}{2\hbar^2}}, \quad (22)$$

respectively. The Schrödinger normalization constant C_{nl} can be easily found from Appendix B as

$$C_{nl} = \sqrt{\frac{4a\lambda_{nl}n!(n+l+\lambda_{nl}+1)\Gamma(n+2l+2\lambda_{nl}+2)}{(n+1+1)\Gamma(n+2l+2)\Gamma(n+2\lambda_{nl}+1)}}. \quad (23)$$

In Table 2, we calculate some approximation and numerical energy eigenvalues of the Schrödinger equation with Yukawa potential for arbitrary values of radial and orbital n and l quantum numbers, respectively. In our numerical calculations, we used the parameter values $m_1 = 5 \text{ fm}^{-1}$, $m_2 = 5 \text{ fm}^{-1}$, $V_0 = 1$ and $a = (0.01, 0.005, 0.001) \text{ fm}^{-1}$.

Table 2. Approximation and numerical binding energy spectrum of the Schrödinger equation with Yukawa potential for various values of n and l quantum numbers.

		$-E_{nl}$					
		$a = 0.01 \text{ fm}^{-1}$		$a = 0.01 \text{ fm}^{-1}$		$a = 0.01 \text{ fm}^{-1}$	
n	l	Approximation	Numerical	Approximation	Numerical	Approximation	Numerical
1	0	0.2926	0.2944	0.3025	0.3019	0.3105	0.3103
2	0	0.1191	0.1229	0.1289	0.1287	0.1369	0.1342
	1	0.0584	0.0577	0.0682	0.0684	0.0761	0.0762
3	0	0.0584	0.0680	0.0682	0.0731	0.0761	0.0778
	1	0.0305	0.0334	0.0401	0.0413	0.0480	0.0475
	2	0.0154	0.0155	0.0249	0.0242	0.0327	0.0320
4	0	0.0305	0.0419	0.0401	0.0467	0.0480	0.0509
	1	0.0154	0.0214	0.0249	0.0277	0.0327	0.0330
	2	0.0065	0.0095	0.0157	0.0165	0.0235	0.0229
	3	0.0008	0.0010	0.0098	0.0099	0.0175	0.0178
5	0	0.0154	0.0274	0.0249	0.0318	0.0327	0.0359
6	0	0.0065	0.0084	0.0157	0.0226	0.0235	0.0264
7	0	0.0008	0.0011	0.0098	0.0165	0.0175	0.0202

Second, upon inserting $a = 0$ in Eq. (21), the bound-state energy formula of two particles interacting via the Coulomb potential within the Schrödinger equation is

$$E_{nl} = -\frac{\mu}{2\hbar^2} \frac{V_0^2}{(n+l+1)^2}. \quad (24)$$

4. Final Remarks and Conclusion

We have presented approximate analytical bound-state solutions of the two-body SS equation with the Yukawa potential for arbitrary l -states by considering an approximation to the centrifugal term. The semirelativistic bound-state energy eigenvalues and their corresponding normalized wave functions are obtained by the parametric generalization of the NU method. It is found that wave function solution can be expressed in terms of the Jacobi polynomials. The results obtained from exact numerical solution of Eq. (6) are in good agreement with the approximate solution of the quantum system in Eq. (14) for low screening region (short-range potential).

The aim of solving the Yukawa potential returns to the following reasons: First, in the low screening region where the screening parameter a is small (i.e. $a \ll 1$), the potential reduces to the Killingbeck potential,⁵⁴⁻⁵⁶ i.e. $V(r) = ar^2 + br - c/r$, where a , b and c are potential constants that can be obtained after making expansion to the Yukawa potential. Second, it can also be reduced into the Cornell potential,^{57,58} i.e. $V(r) = br - c/r$. These two potentials are usually used in the study of mesons and baryons. Third, when the screening parameter approaches to zero, the Yukawa potential turns to become the Coulomb potential.

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Appendix A. Parametric Generalization of the NU Method

The NU method is used to solve second order differential equations with an appropriate coordinate transformation $s = s(r)$ ⁴⁷

$$\psi_n''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi_n'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi_n(s) = 0, \quad (A.1)$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most of second degree, and $\tilde{\tau}(s)$ is a first-degree polynomial. To make the application of the NU method simpler and direct without need to check the validity of solution, we present a shortcut for the method. So, at first we write the general form of the Schrödinger-like equation (A.1) in a more general form applicable to any potential as follows⁵⁹⁻⁶¹

$$\psi_n''(s) + \left(\frac{c_1 - c_2s}{s(1 - c_3s)} \right) \psi_n'(s) + \left(\frac{-As^2 + Bs - C}{s^2(1 - c_3s)^2} \right) \psi_n(s) = 0, \quad (A.2)$$

satisfying the wave functions

$$\psi_n(s) = \phi(s)y_n(s). \quad (\text{A.3})$$

Comparing (B.2) with its counterpart (B.1), we obtain the following identifications:

$$\tilde{\tau}(s) = c_1 - c_2s, \quad \sigma(s) = s(1 - c_3s), \quad \tilde{\sigma}(s) = -As^2 + Bs - C. \quad (\text{A.4})$$

Following the NU method,⁴⁷ we obtain the following,⁵⁹⁻⁶¹

(i) the relevant constant:

$$\begin{aligned} c_4 &= \frac{1}{2}(1 - c_1), & c_5 &= \frac{1}{2}(c_2 - 2c_3), \\ c_6 &= c_5^2 + A, & c_7 &= 3c_4c_5 - B, \\ c_8 &= c_4^2 + C, & c_9 &= c_3(c_7 + c_3c_8) + c_6, \\ c_{10} &= c_1 + 2c_4 + 2\sqrt{c_8} - 1 > -1, & c_{11} &= 1 - c_1 - 2c_4 + \frac{2}{c_3}\sqrt{c_9} > -1, \quad c_3 \neq 0, \\ c_{12} &= c_4 + \sqrt{c_8} > 0, & c_{13} &= -c_4 + \frac{1}{c_3}(\sqrt{c_9} - c_5) > 0, \quad c_3 \neq 0. \end{aligned} \quad (\text{A.5})$$

(ii) the essential polynomial functions:

$$\pi(s) = c_4 + c_5s - [(\sqrt{c_9} + c_3\sqrt{c_8})s - \sqrt{c_8}], \quad (\text{A.6})$$

$$k = -(c_7 + 2c_3c_8) - 2\sqrt{c_8c_9}, \quad (\text{A.7})$$

$$\tau(s) = c_1 + 2c_4 - (c_2 - 2c_5)s - 2[(\sqrt{c_9} + c_3\sqrt{c_8})s - \sqrt{c_8}], \quad (\text{A.8})$$

$$\tau'(s) = -2c_3 - 2(\sqrt{c_9} + c_3\sqrt{c_8}) < 0. \quad (\text{A.9})$$

(iii) The energy equation:

$$c_2n - (2n + 1)c_5 + (2n + 1)(\sqrt{c_9} + c_3\sqrt{c_8}) + n(n - 1)c_3 + c_7 + 2c_3c_8 + 2\sqrt{c_8c_9} = 0. \quad (\text{A.10})$$

(iv) The wave functions

$$\rho(s) = s^{c_{10}}(1 - c_3s)^{c_{11}}, \quad (\text{A.11})$$

$$\phi(s) = s^{c_{12}}(1 - c_3s)^{c_{13}}, \quad c_{12} > 0, \quad c_{13} > 0, \quad (\text{A.12})$$

$$y_n(s) = P_n^{(c_{10}, c_{11})}(1 - 2c_3s), \quad c_{10} > -1, \quad c_{11} > -1, \quad (\text{A.13})$$

$$\psi_{n\kappa}(s) = N_{n\kappa}s^{c_{12}}(1 - c_3s)^{c_{13}}P_n^{(c_{10}, c_{11})}(1 - 2c_3s), \quad (\text{A.14})$$

where $P_n^{(\mu, \nu)}(x)$, $\mu > -1$, $\nu > -1$, and $x \in [-1, 1]$ are Jacobi polynomials with

$$P_n^{(\alpha, \beta)}(1 - 2s) = \frac{(\alpha + 1)_n}{n!} {}_2F_1(-n, 1 + \alpha + \beta + n; \alpha + 1; s), \quad (\text{A.15})$$

and $N_{n\kappa}$ is a normalization constant. Also, the above wave functions can be expressed in terms of the hypergeometric function as

$$\psi_{n\kappa}(s) = N_{n\kappa}s^{c_{12}}(1 - c_3s)^{c_{13}} {}_2F_1(-n, 1 + c_{10} + c_{11} + n; c_{10} + 1; c_3s), \quad (\text{A.16})$$

where $c_{12} > 0$, $c_{13} > 0$ and $s \in [0, 1/c_3]$, $c_3 \neq 0$.

Appendix B. Calculation of the Normalization Constant in the SS Equation

To compute the normalization constant A_{nl} in Eq. (19), it is easy to show with the use of $R_{nl}(r) = r^{-1}\psi_{nl}(r)$, that

$$\int_0^\infty |R_{nl}(r)|^2 r^2 dr = \int_0^\infty |\psi_{nl}(r)|^2 dr = \int_0^1 |\psi_{nl}(s)|^2 \frac{ds}{2\alpha s} = 1, \quad (\text{B.1})$$

where we have used the substitution $s = e^{-2ar}$. Inserting Eq. (18) into (B.1) and using the following definition of the Jacobi polynomials⁶²

$$P_n^{(p_0, w_0)}(1-2s) = \frac{\Gamma(n+p_0+1)}{n!\Gamma(p_0+1)} {}_2F_1(-n, p_0+w_0+n+1; 1+p_0; s), \quad (\text{B.2})$$

where $p_0 = 2\varepsilon_{nl}$ and $w_0 = 2v$, we arrive at

$$\begin{aligned} |A_{nl}|^2 & \int_0^1 s^{2\varepsilon_{nl}-1} (1-s)^{2v+1} \{ {}_2F_1(-n, 2\varepsilon_{nl}+2v+1+n; 1+2\varepsilon_{nl}; s) \}^2 ds \\ & = 2a \left(\frac{n!\Gamma(2\varepsilon_{nl}+1)}{\Gamma(n+2\varepsilon_{nl}+1)} \right)^2. \end{aligned} \quad (\text{B.3})$$

where ${}_2F_1$ is the hypergeometric function. Using, the following integral formula^{63,64}

$$\begin{aligned} & \int_0^1 z^{2\lambda-1} (1-z)^{2(\eta+1)} \{ {}_2F_1(-n, 2(\lambda+\eta+1)+n; 1+2\lambda+1; z) \}^2 dz \\ & = \frac{(n+\eta+1)n!\Gamma(n+2\eta+2)\Gamma(2\lambda)\Gamma(2\lambda+1)}{(n+\eta+\lambda+1)\Gamma(n+2\lambda+1)\Gamma(n+2\lambda+2\eta+2)}, \quad \eta > -\frac{3}{2}, \quad \lambda > 0, \end{aligned} \quad (\text{B.4})$$

we can get the normalization constant as

$$A_{nl} = \sqrt{\frac{2a\varepsilon_{nl}n!(2n+2v+2\varepsilon_{nl}+1)\Gamma(n+2v+2\varepsilon_{nl}+1)}{(n+v+1/2)\Gamma(n+2v+1)\Gamma(n+2\varepsilon_{nl}+1)}}. \quad (\text{B.5})$$

The relation (B.5) can be used to compute the normalization constant for $n = 0, 1, 2, \dots$ for the ground state ($n = 0$), we have

$$A_{0l} = \sqrt{\frac{a(2v+2\varepsilon_{nl}+1)}{(v+1/2)B(2\varepsilon_{nl}, 2v+1)}}, \quad (\text{B.6})$$

where

$$B(2\varepsilon_{nl}, 2v+1) = \frac{\Gamma(2v+1)\Gamma(1+2\varepsilon_{nl})}{2\varepsilon_{nl}\Gamma(2v+1+2\varepsilon_{nl})}. \quad (\text{B.7})$$

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