Area Moment of Inertia

A-1 Introduction

Calculation the moment of distributed forces.

Examples of distributed forces: Pressure, stress

**Pressure**

\[ dM_{\text{AB}} = py \, dA \]
\[ = ky^2 \, dA \]

\[ M = k \int y(ydA) \]

\[ M = k \int y^2 \, dA \]

**Bending Stress**

\[ dM_{\text{OO}} = \sigma y \, dA \]
\[ = ky^2 \, dA \]

\[ M = k \int y^2 \, dA \]
Torsion

The total moment involves an integral of the form:

\[ \int (\text{distance})^2 d(\text{area}) \]

\[ I = \int y(ydA) = \int y^2 dA \]

This integral is called *moment of inertia of an area*

or more fitting:

The *second moment of area*, since the first moment \( ydA \) is multiplied by the moment arm \( y \) to obtain the second moment for the element \( dA \).

(Centroid; First moment of area)

The moment of inertia of an area is a purely mathematical property of the area and in *itself* has no physical significance.
A-2 Definitions

Rectangular and polar moments of inertia

The moments of inertia of the element $dA$ about the $x$- and $y$- axis are:

$$dI_x = y^2 dA$$
$$dI_y = x^2 dA,$$

The rectangular moment of inertia of the whole $A$ area about the same axis are:

$$I_x = \int y^2 dA$$
$$I_y = \int x^2 dA$$

Note: All parts of the element $dA$ must have the same distance from the axis of rotation.
The *polar* moment of inertia

The polar moment of inertia of \( dA \) about \( z \)-axis:

\[
dl_z = r^2 dA
\]

The polar moment of inertia of the entire area about the \( z \)-axis:

\[
I_z = \int r^2 \, dA
\]

Because of

\[
r^2 = x^2 + y^2
\]

We get:

\[
I_z = I_x + I_y
\]

Other symbols: \( J, I_p, I_r \)

The second moment of area is always a *positive* quantity.  
\((x^2, y^2, r^2, \text{square of a distance})\)
Radius of Gyration:

The *radius of gyration* $k$ is a measure of the distribution of the area from the axis of rotation. It is defined as:

$$ k = \sqrt{I/A} $$

Furthermore:

$$ k_x = \sqrt{I_x/A} \quad k_y = \sqrt{I_y/A} \quad k_z = \sqrt{I_z/A} $$

$$ k_z^2 = k_x^2 + k_y^2 $$

For the moments of inertia we get then:

$$ I_x = k_x^2 A \quad I_y = k_y^2 A \quad I_z = k_z^2 A $$
Transfer of Axes

The moment of inertia of an area about a noncentroidal axis may be easily expressed in terms of the moment of inertia about a parallel Centroidal axis.

*Parallel axis theorem* (Steiner Theorem):

\[
I_x = \bar{I}_x + Ad_x^2 \quad I_y = \bar{I}_y + Ad_y^2 \quad I_z = \bar{I}_z + Ad_z^2
\]

Two points in particular should be noted.
- the axes between which the transfer is made *must be parallel*, and
- one of the axes *must pass through the centroid* of the area.